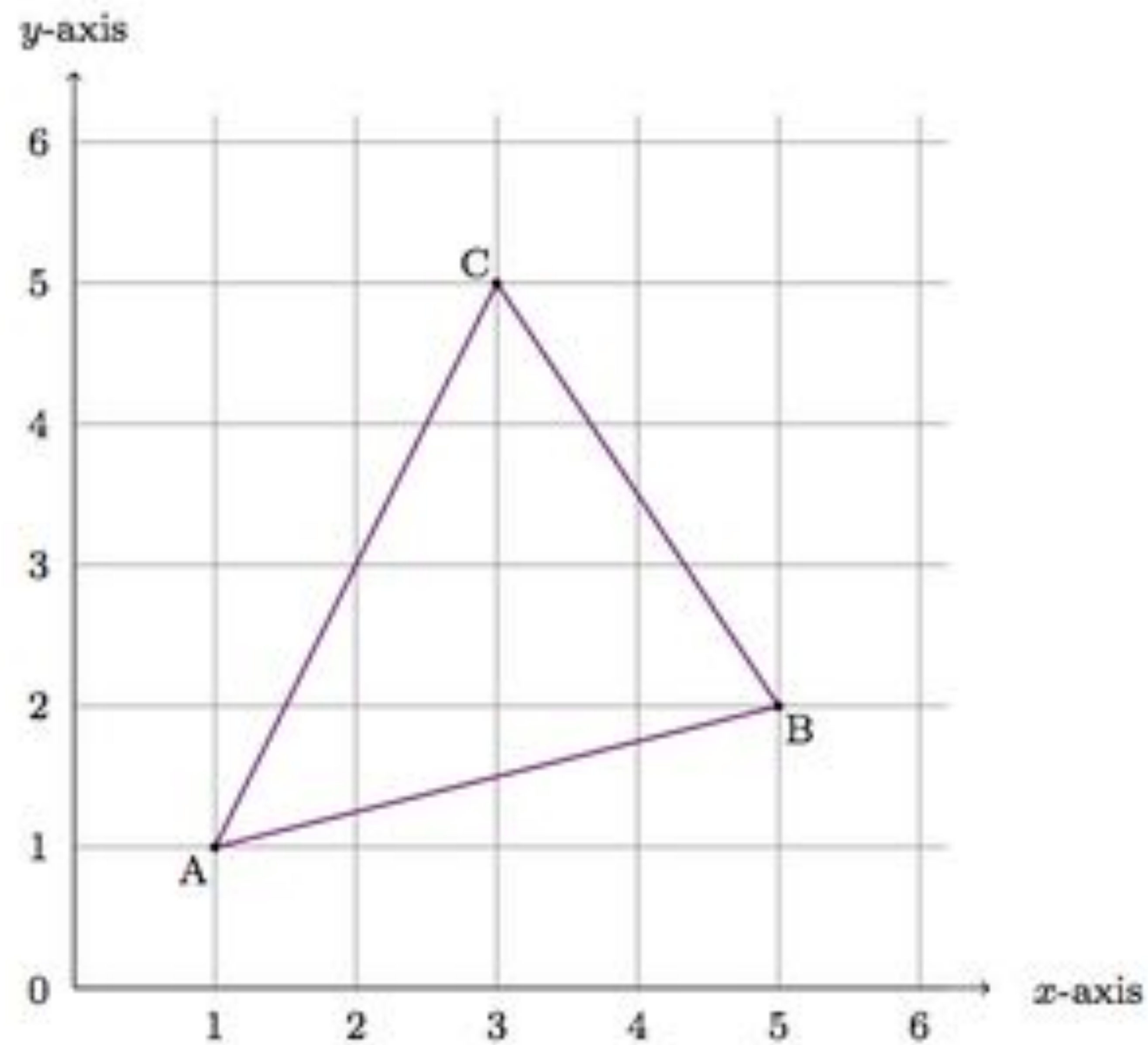
APPLICATIONS OF GEOMETRY AND ALGEBRA: BARYCENTRIC COORDINATES

# COMPUTATIONAL AND APPLIED MATHEMATICS

# WHAT ARE BARYCENTRIC COORDINATES (AND WHY ARE THEY USEFUL)?

# ABSOLUTE VS RELATIVE COORDINATES

- Triangles are usually defined in Cartesian coordinates (*absolute coordinate system*)
- Determining points inside the triangle requires extra calculations



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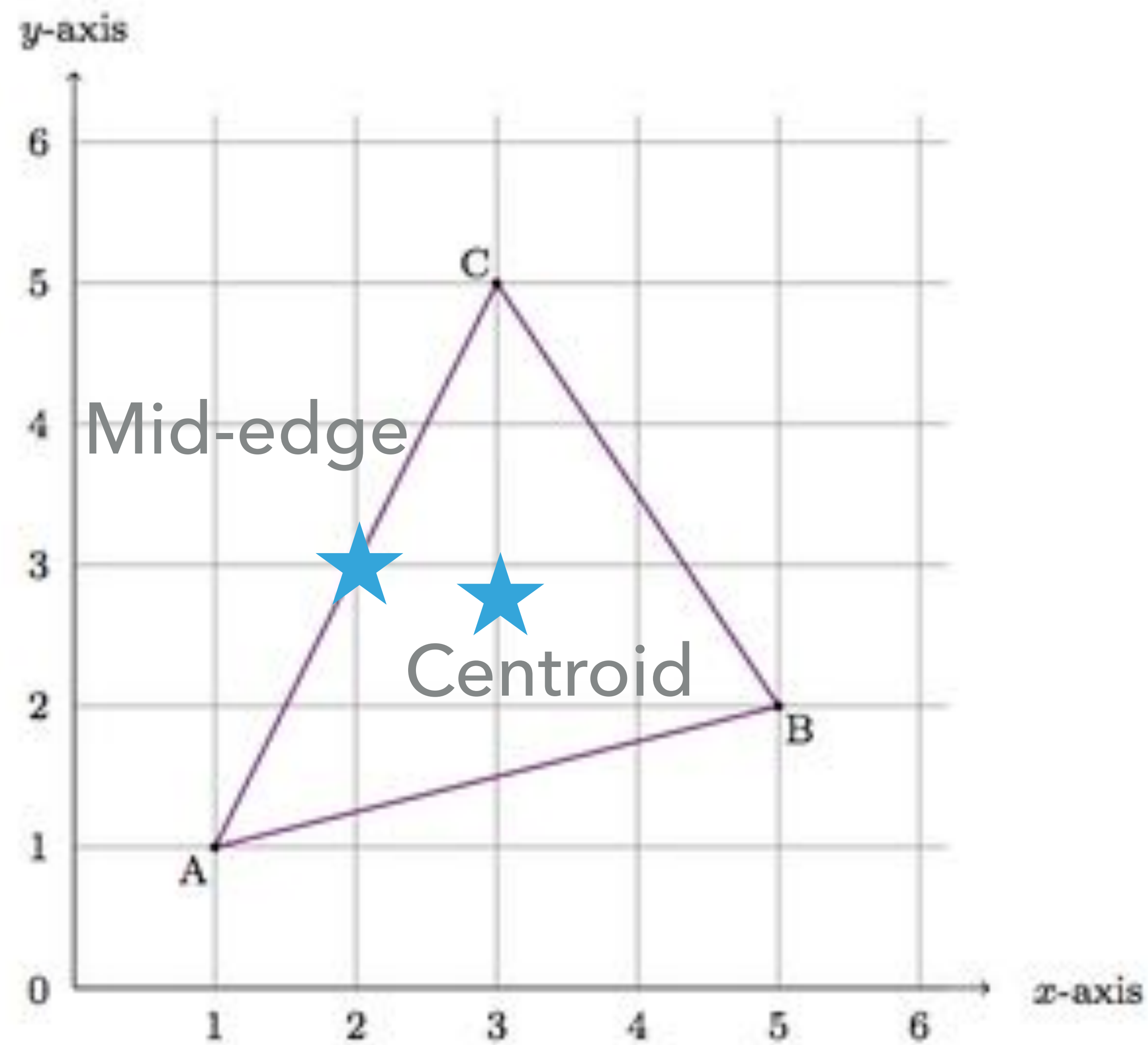



Illustrative Mathematics

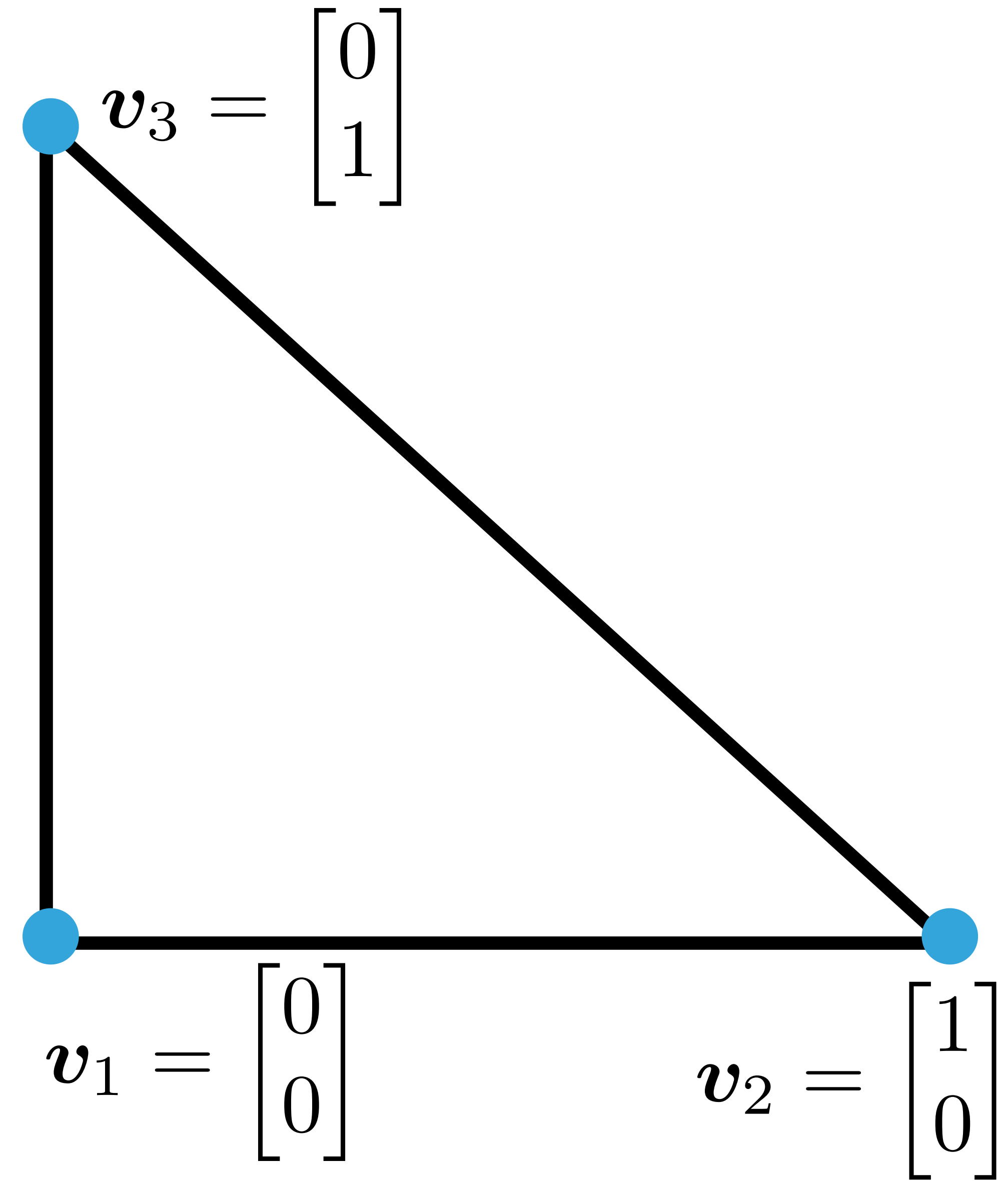
# ABSOLUTE VS RELATIVE COORDINATES

- Barycentric coordinates represent a point inside a triangle *relative* to the triangle's vertices

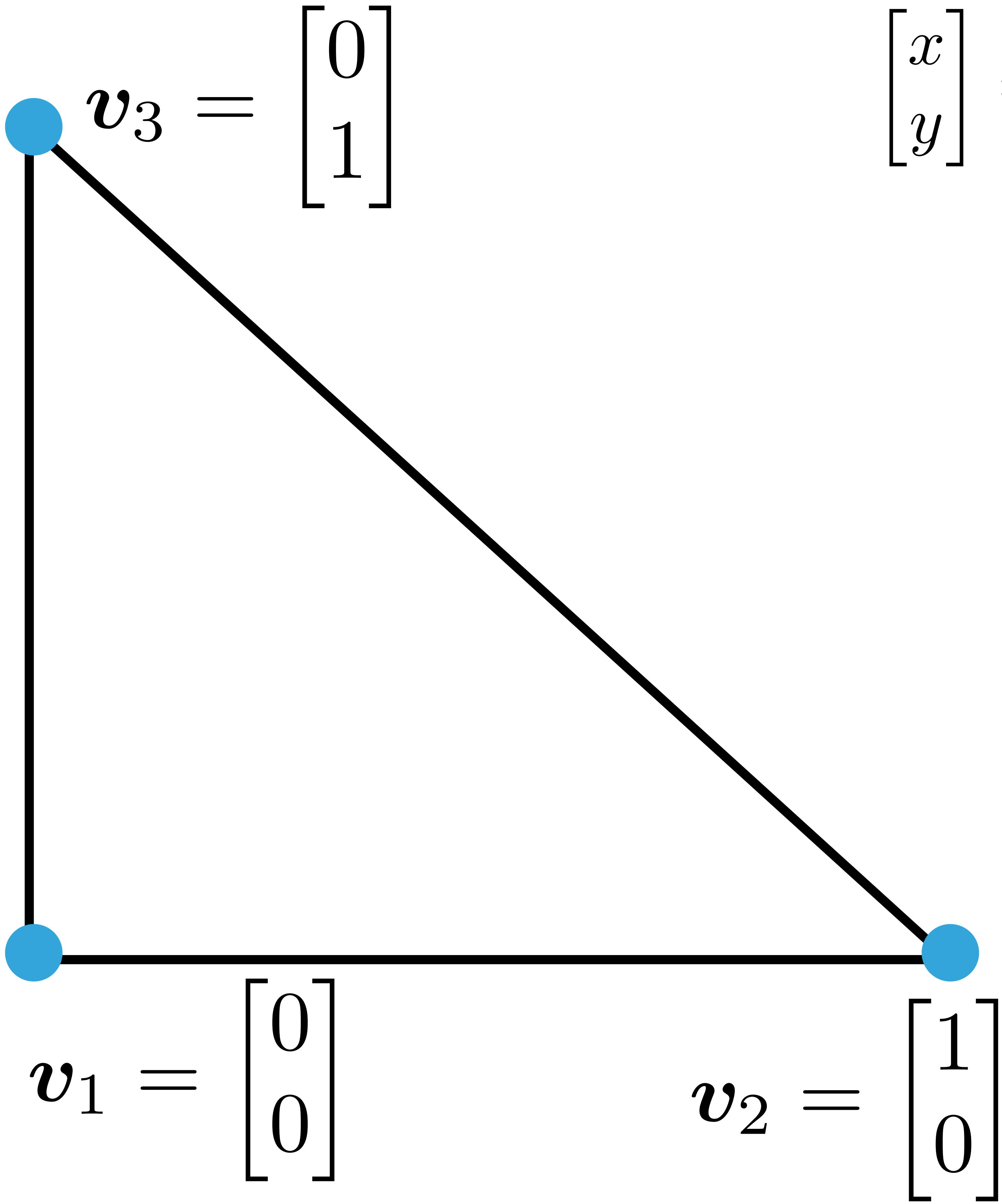
$$\begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \lambda_2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \lambda_3 \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}$$

or

$$\boldsymbol{x} = \lambda_1 \boldsymbol{v}_1 + \lambda_2 \boldsymbol{v}_2 + \lambda_3 \boldsymbol{v}_3$$



$$\begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



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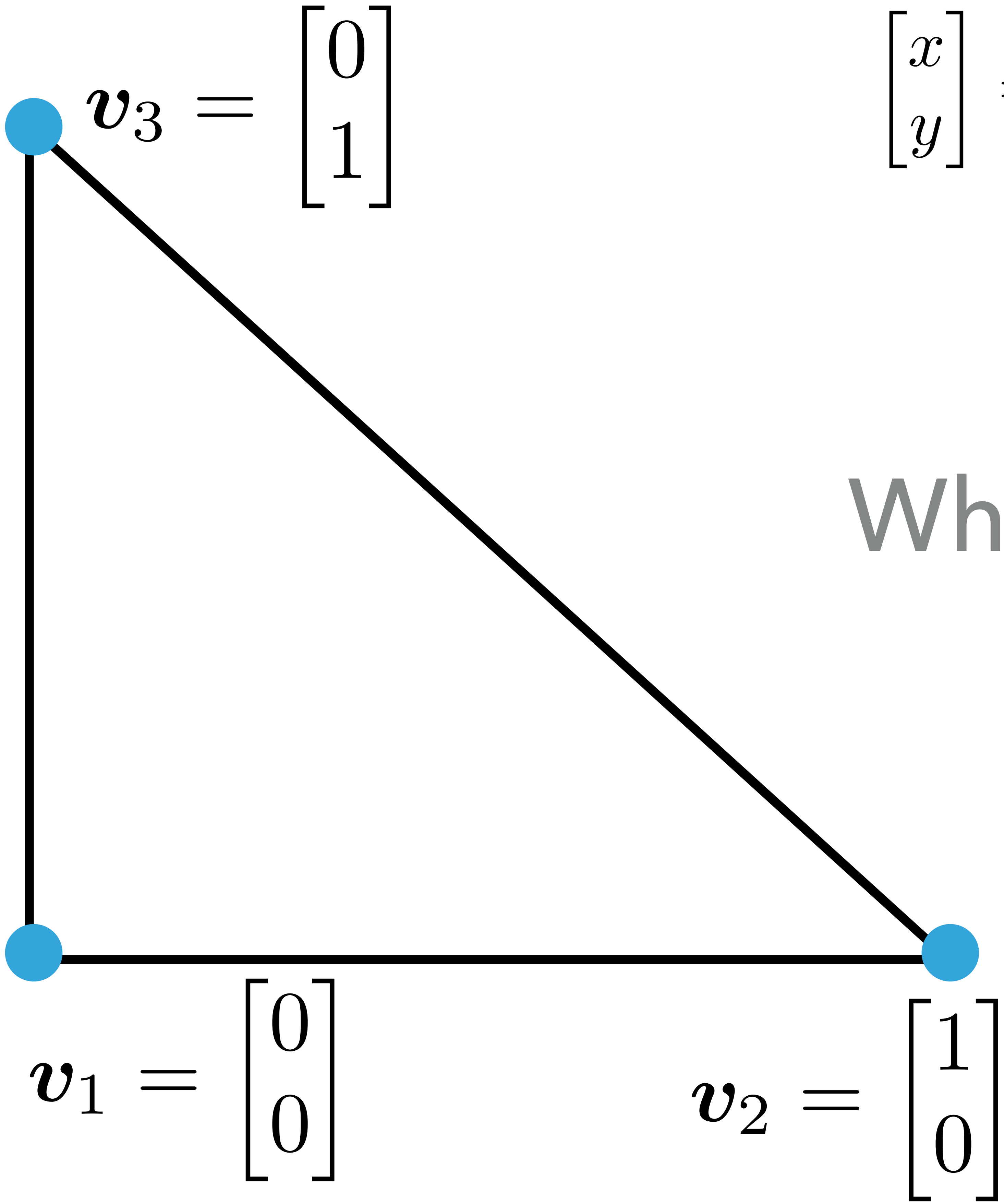


What if

$$(\lambda_1, \lambda_2, \lambda_3) = (1, 0, 0)$$

## BARYCENTRIC COORDINATES

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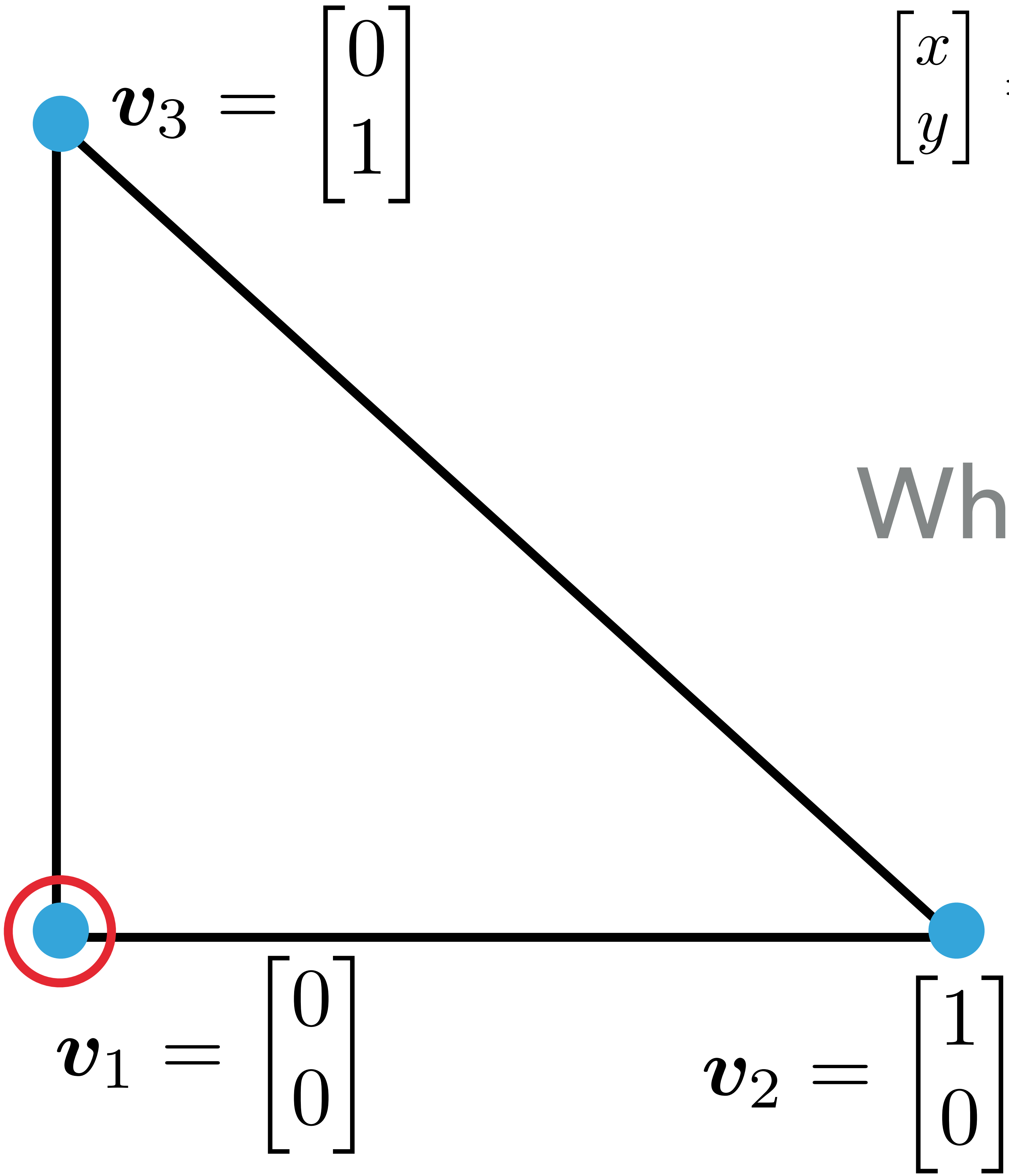


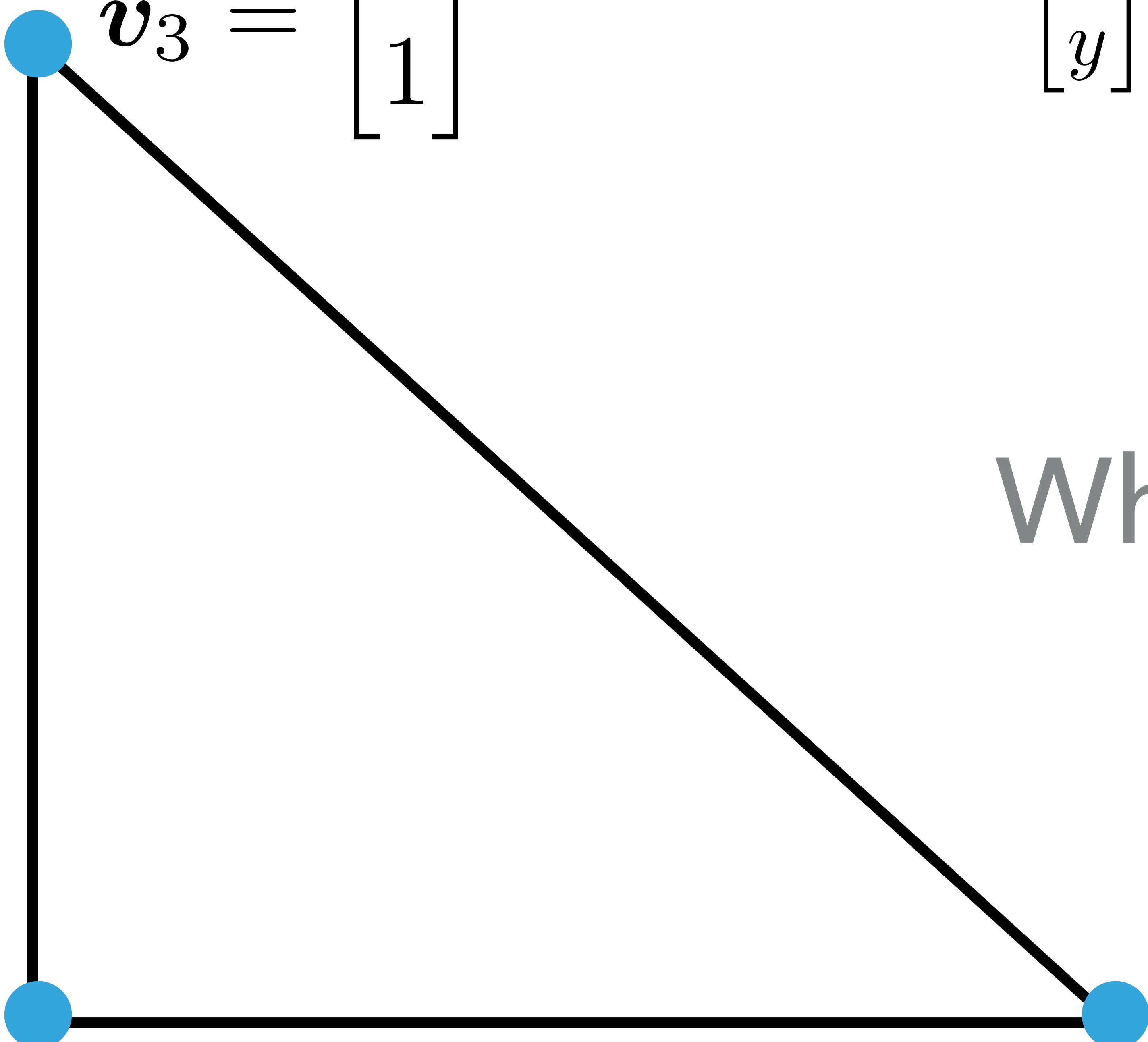

What if

$$(\lambda_1, \lambda_2, \lambda_3) = (1, 0, 0)$$

= first vertex

$$\begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

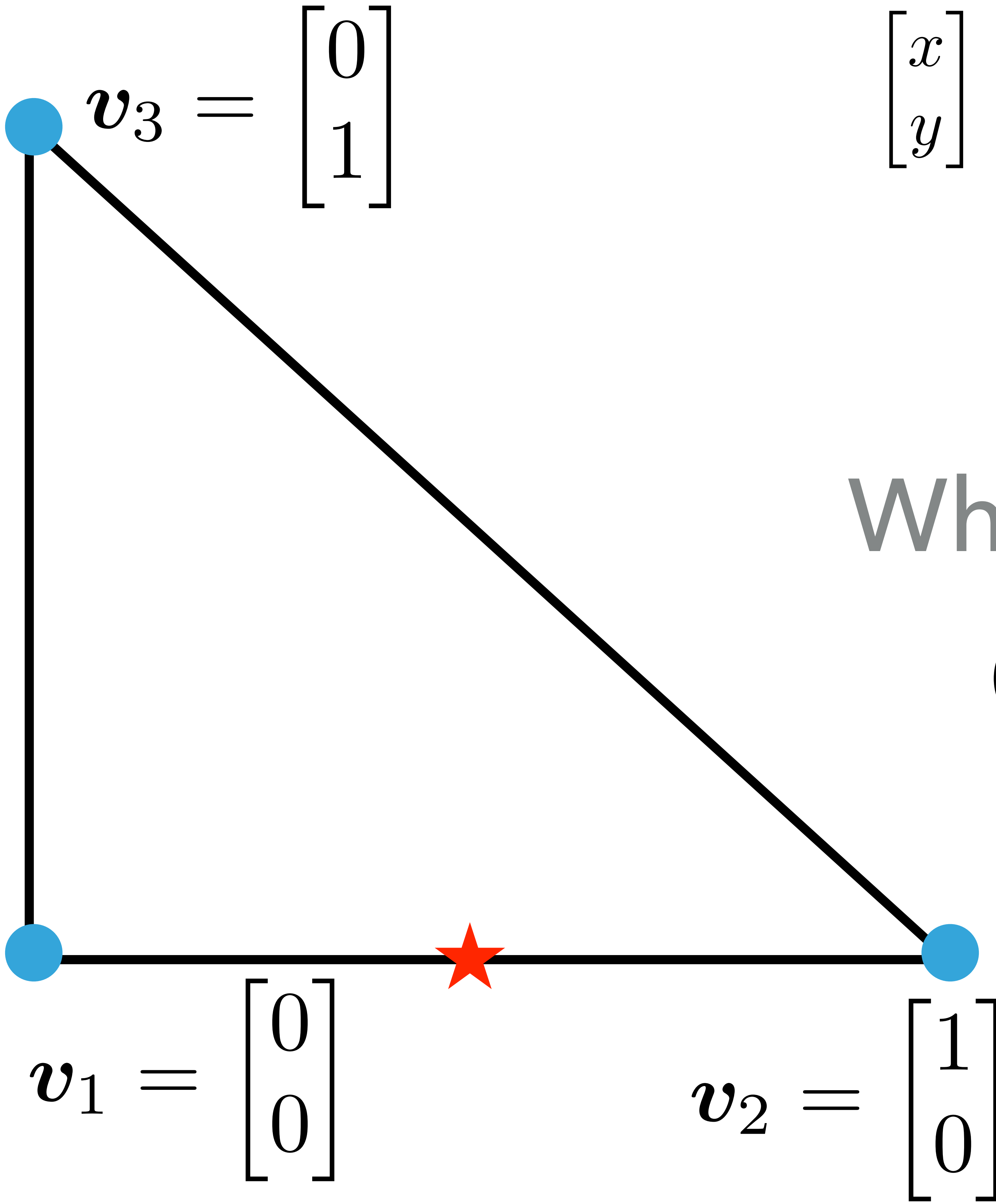


$$\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

What if

$$(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

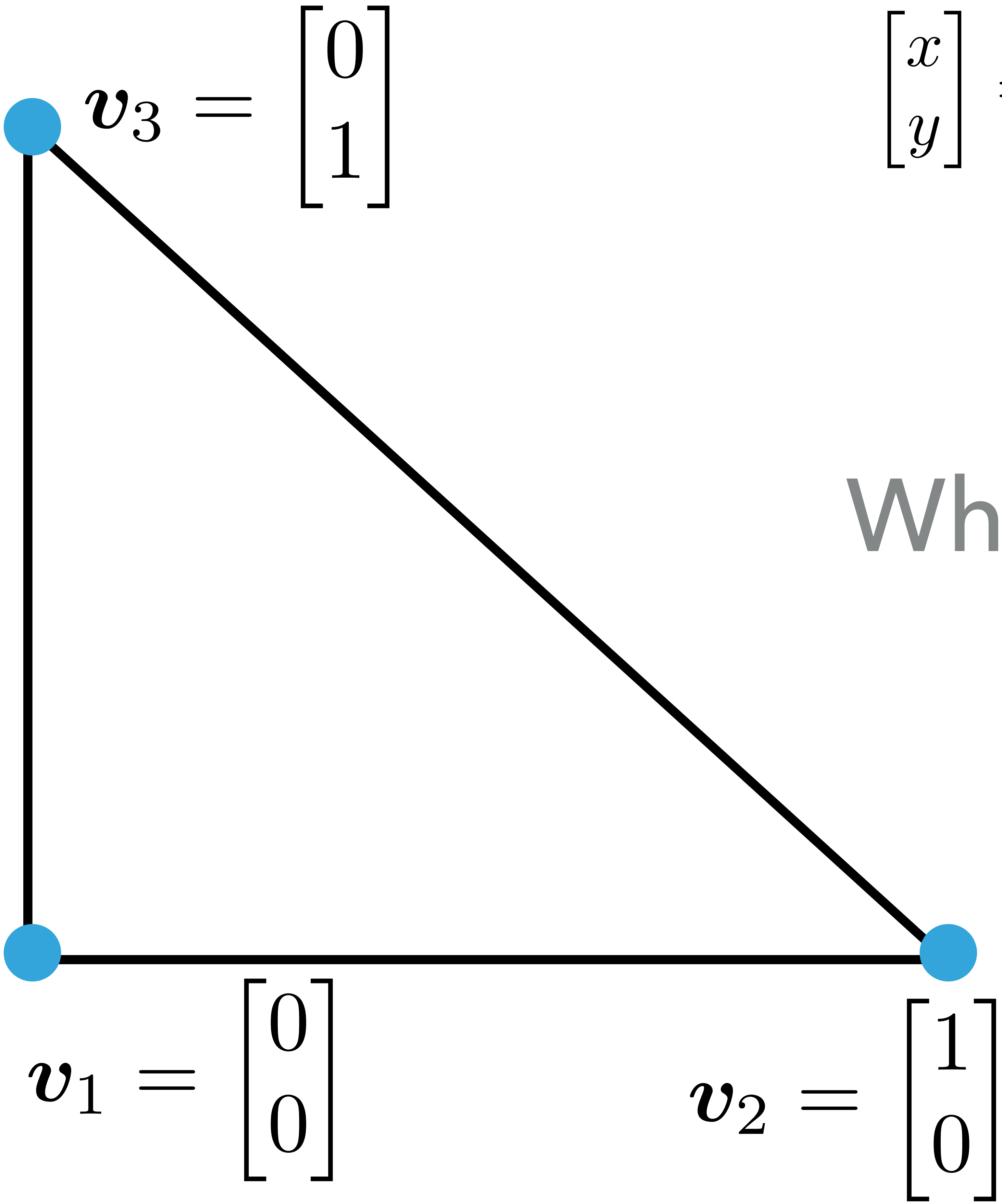
$$\begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$




What if

$$(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

= *midpoint* between first and second vertex

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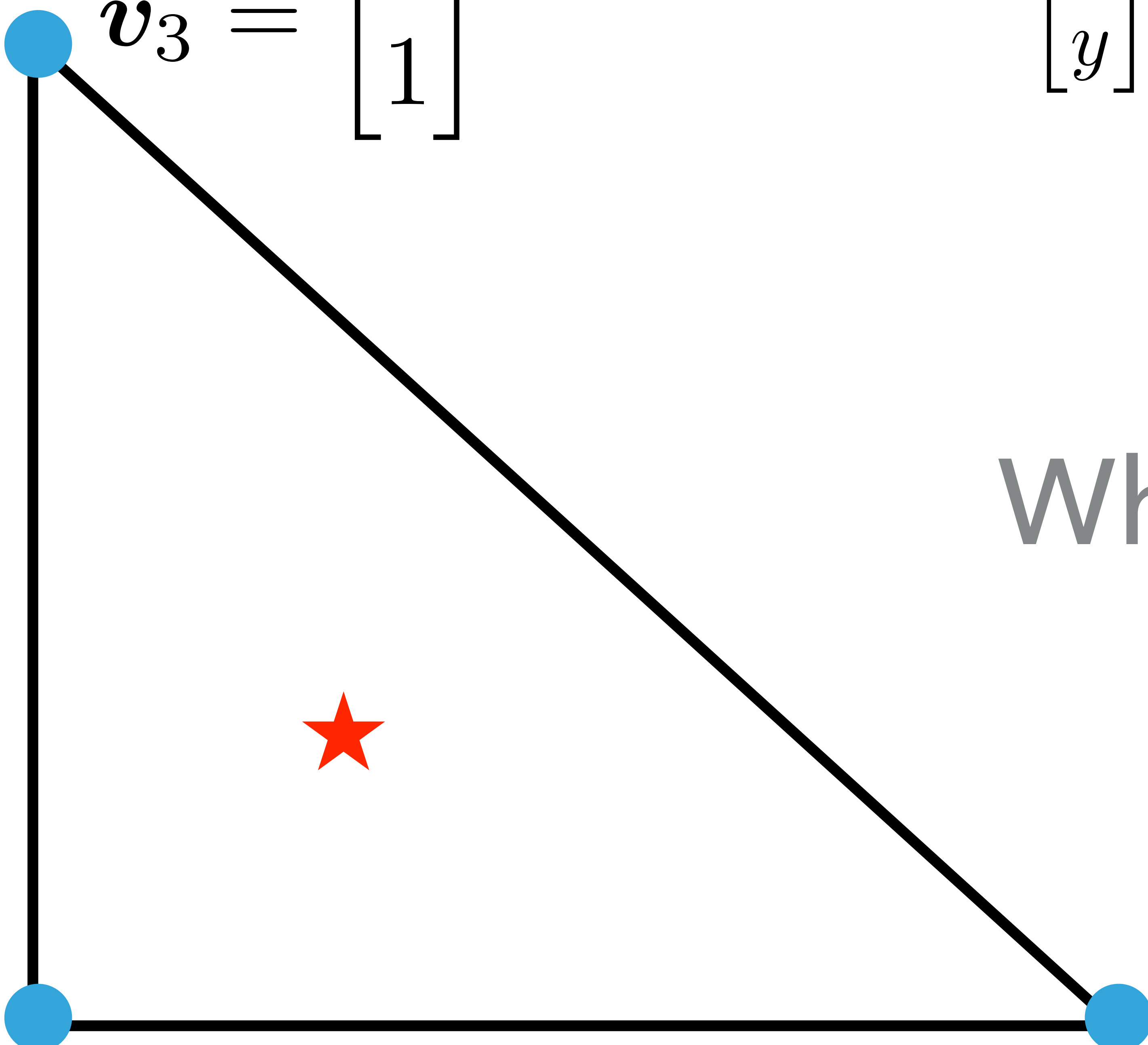


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What if

$$(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

= triangle *centroid*

$$\begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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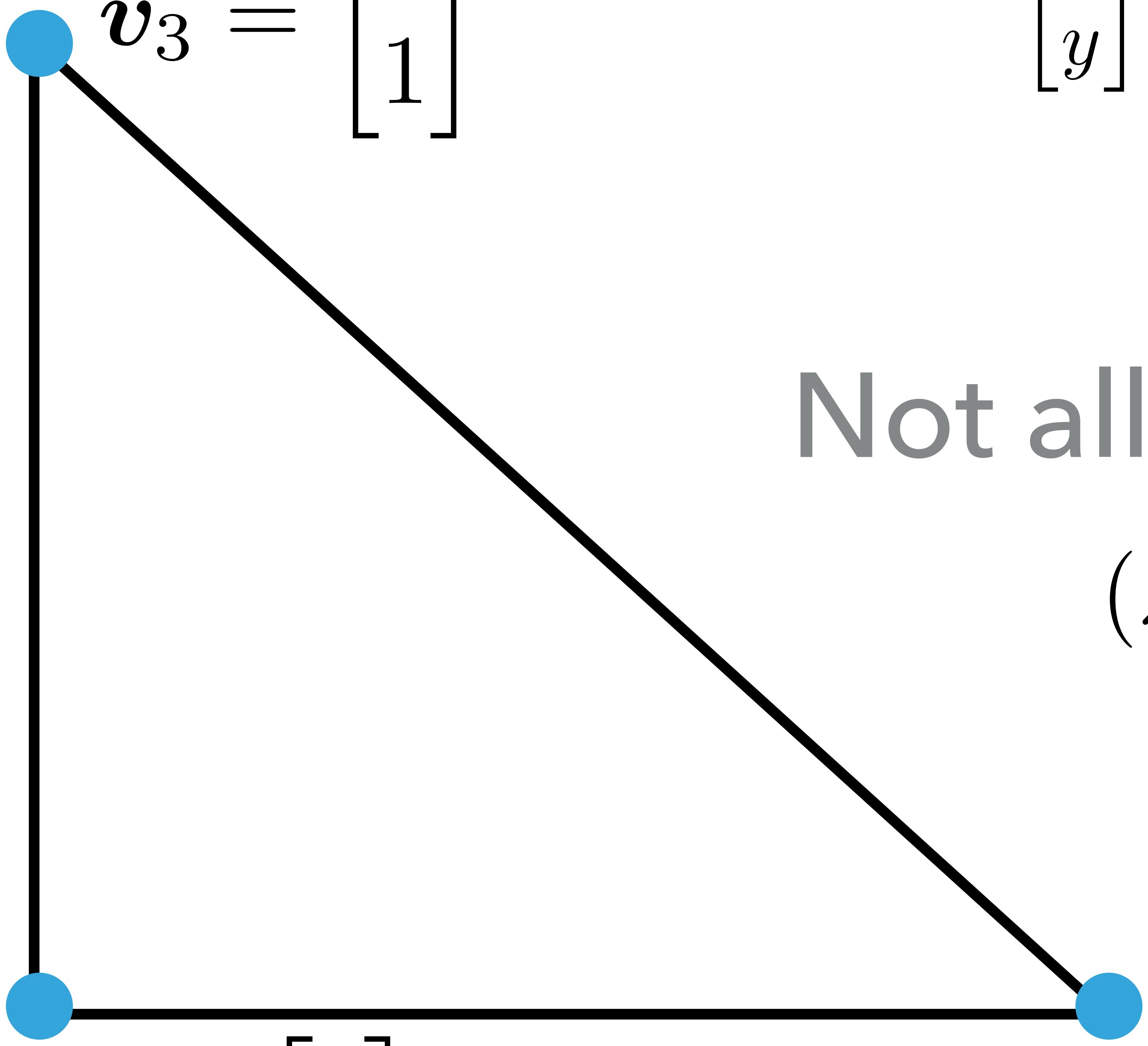
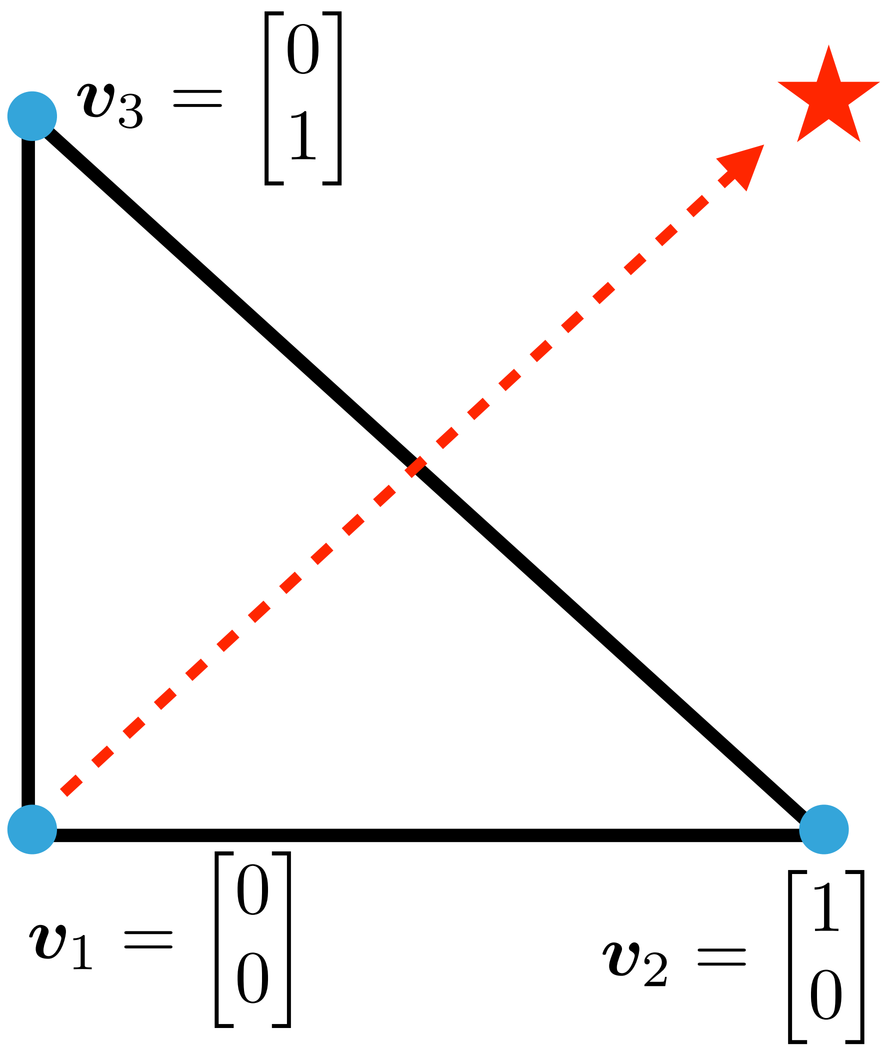


$$\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

## Not all choices work…

$$(\lambda_1, \lambda_2, \lambda_3) = (1, 1, 1)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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# BARYCENTRIC COORDINATES

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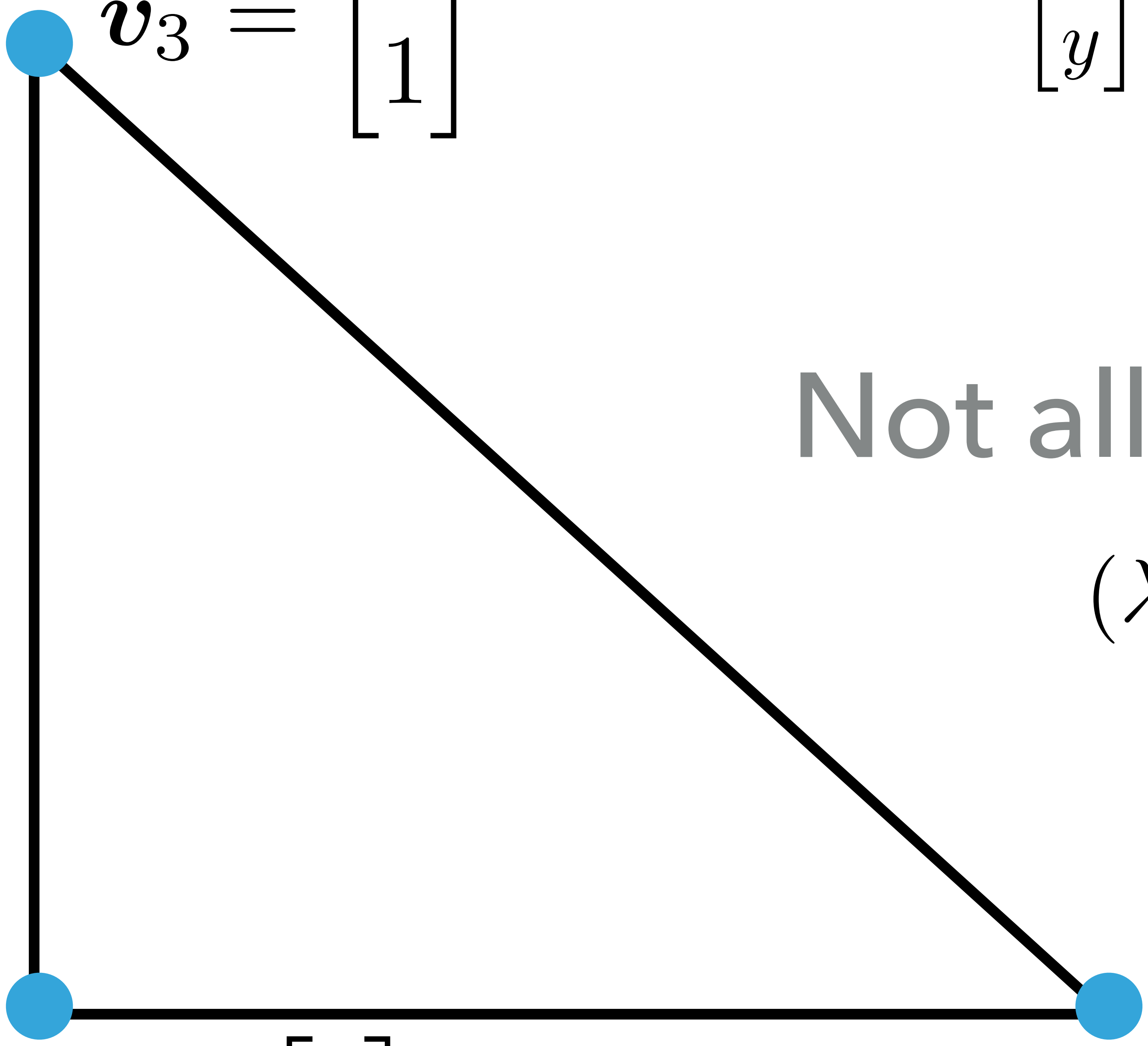

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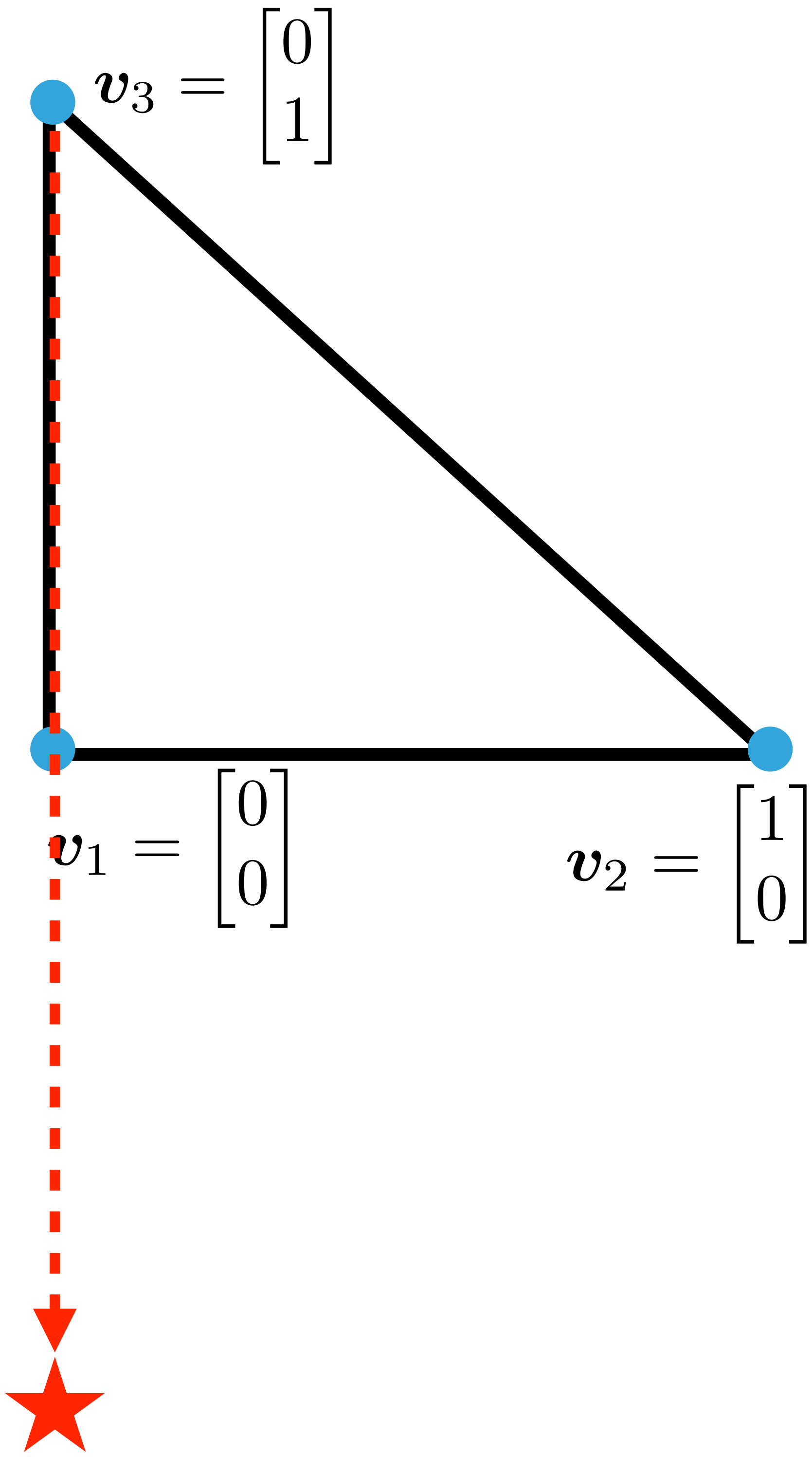


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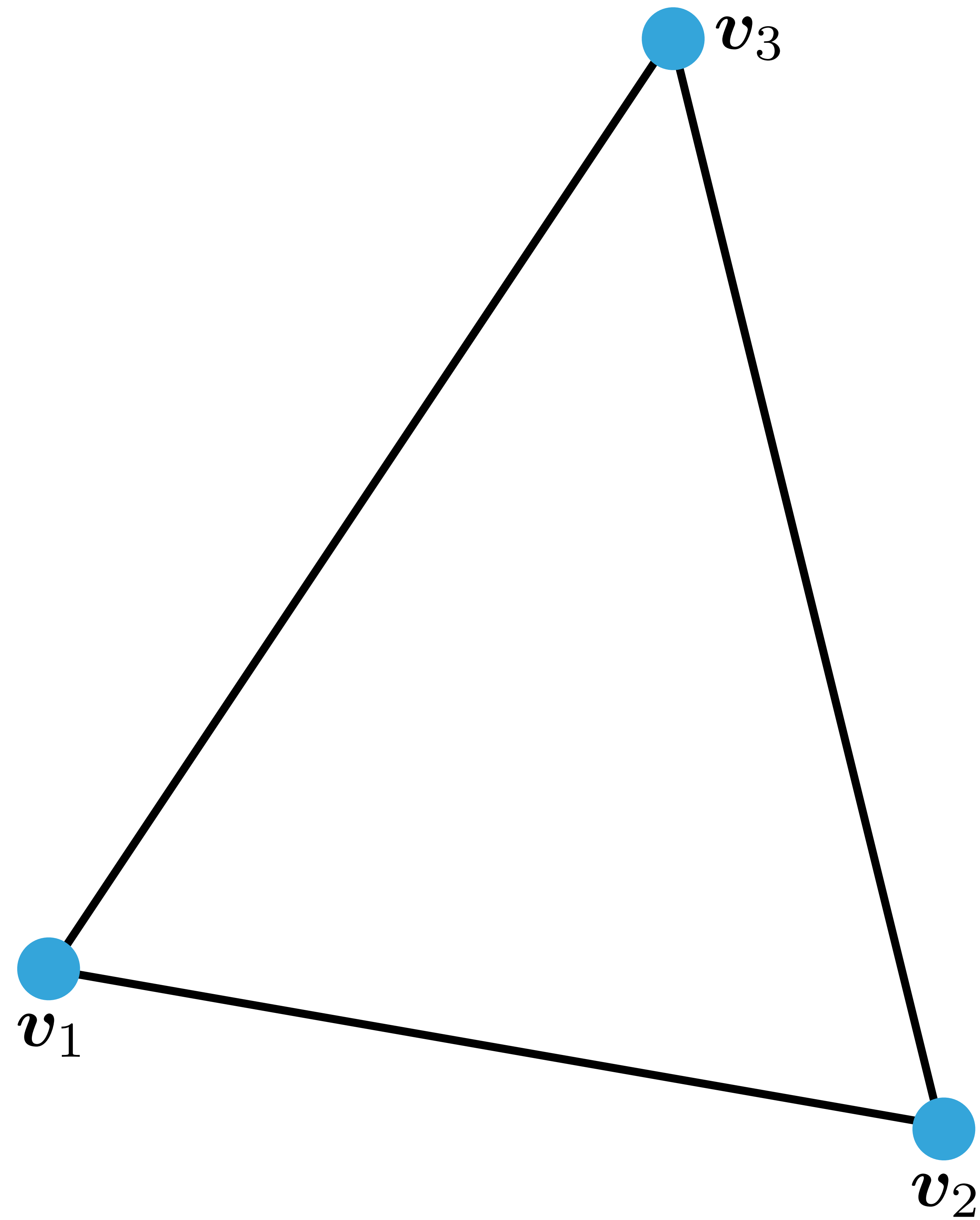
# A LITTLE BIT OF INTUITION

- Barycentric coordinates are also known as *area* coordinates
- Any interior point creates three new triangles
- Barycentric coordinates are also ratios of the **area of a sub-triangle** to the **total triangle area**



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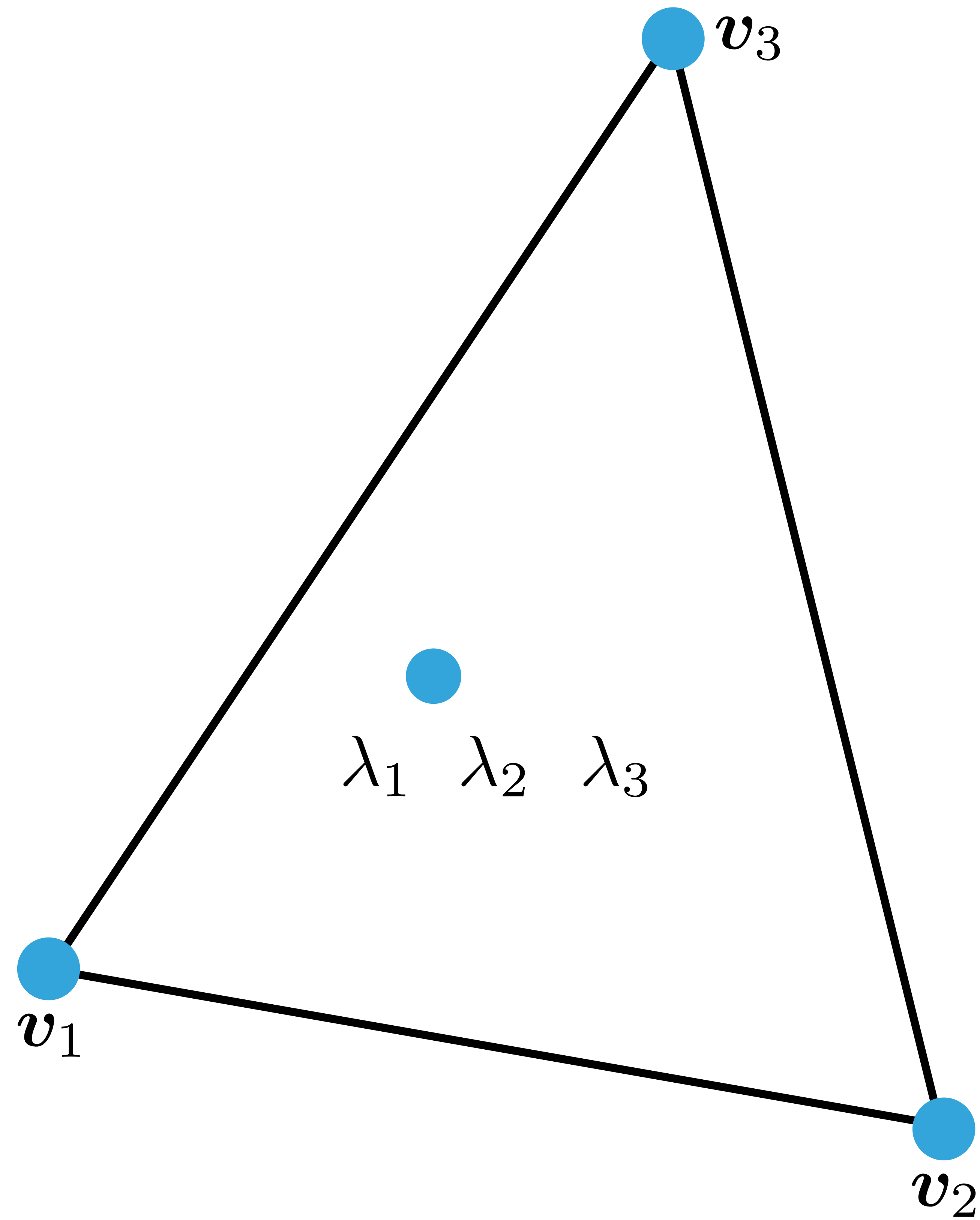

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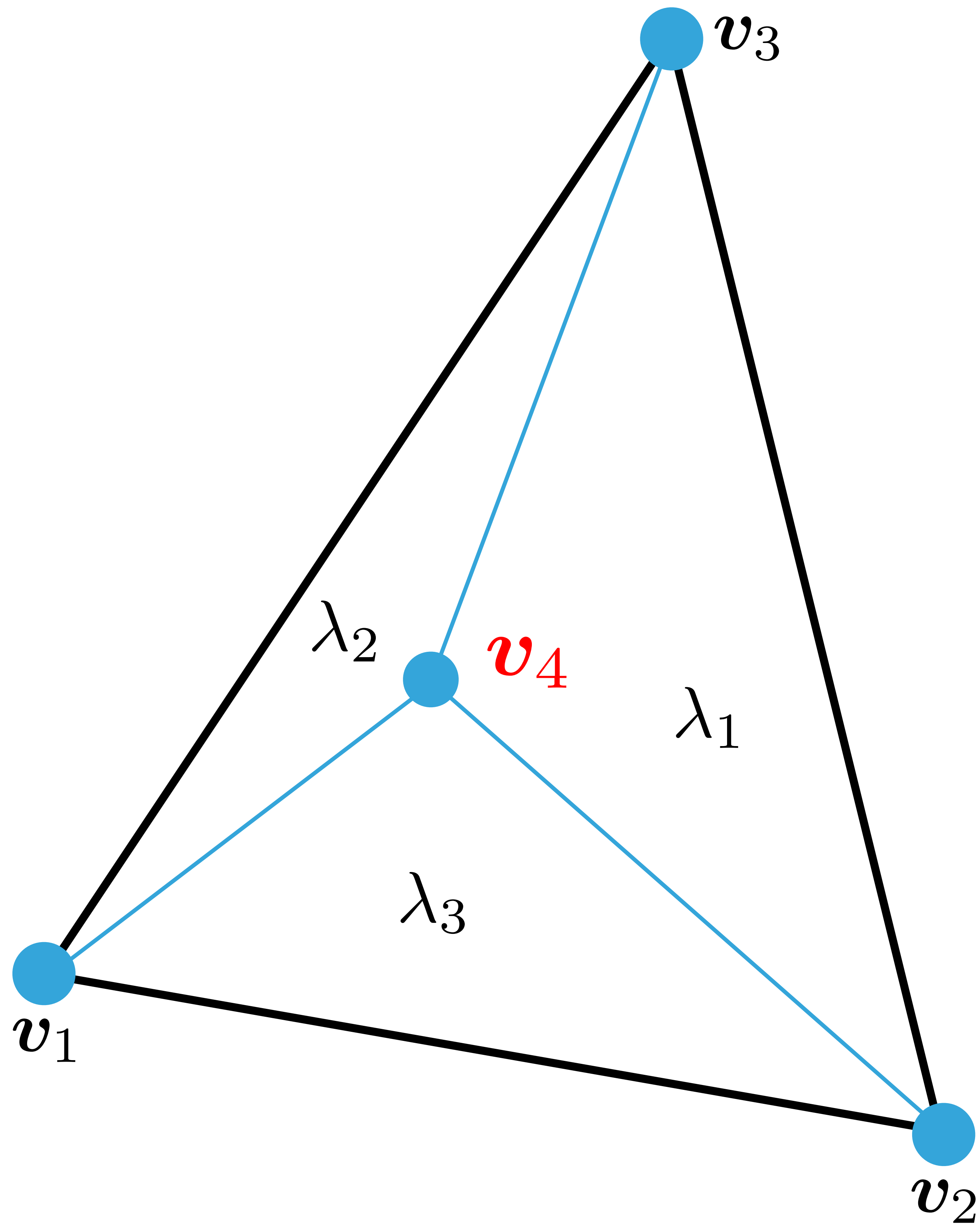

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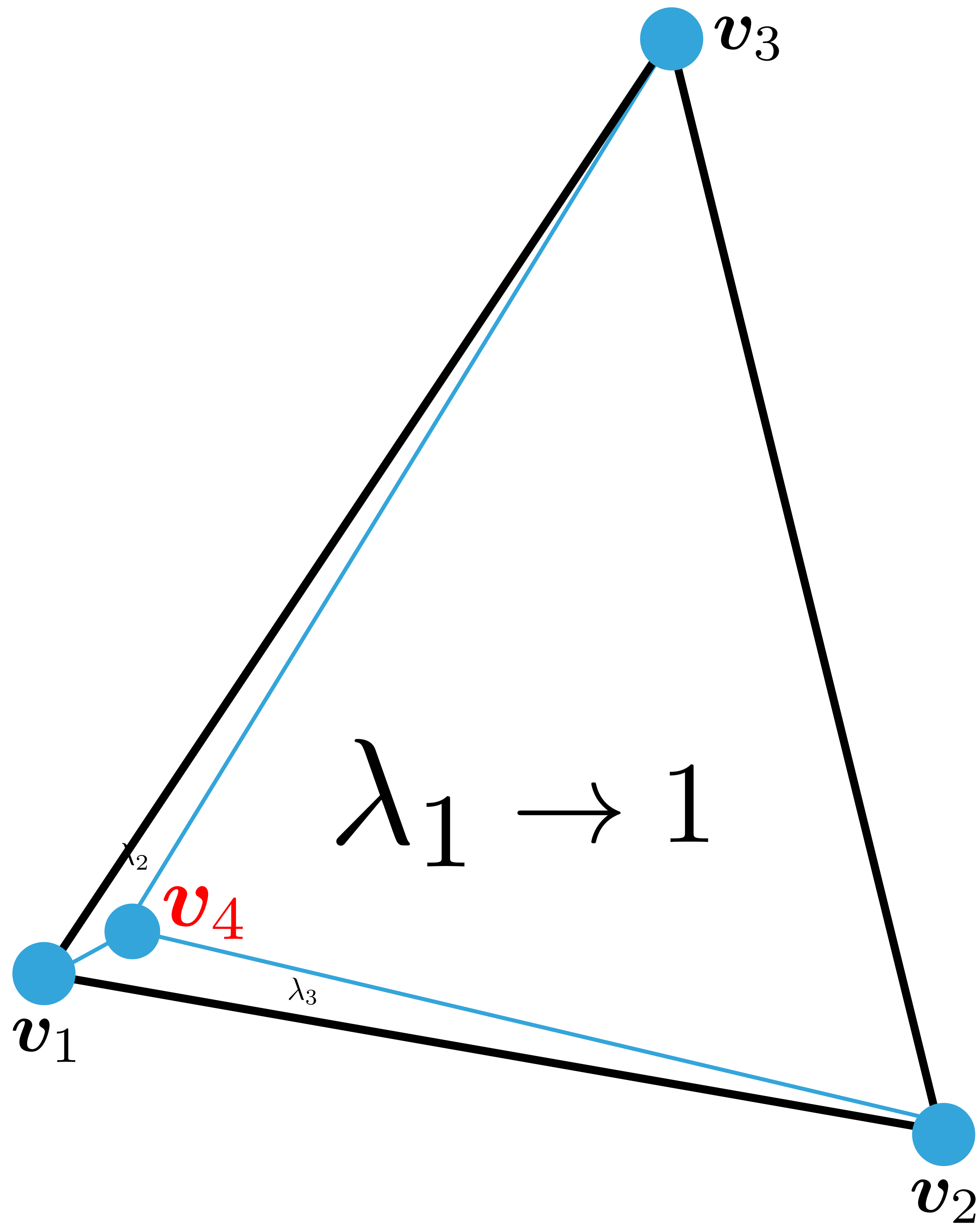

# FROM INTUITION TO EQUATIONS

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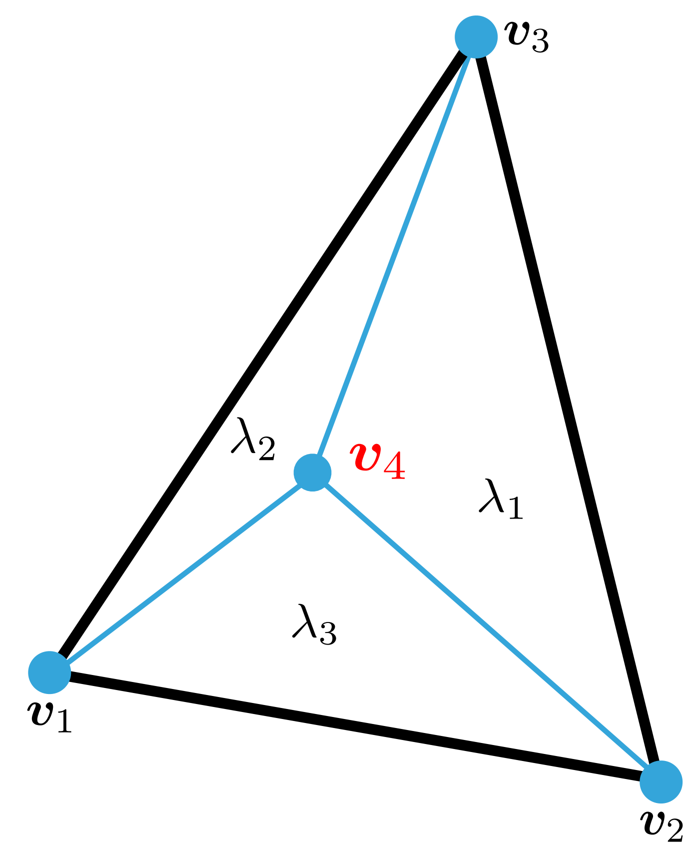


$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$
$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

$$\lambda_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \lambda_2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \lambda_3 \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

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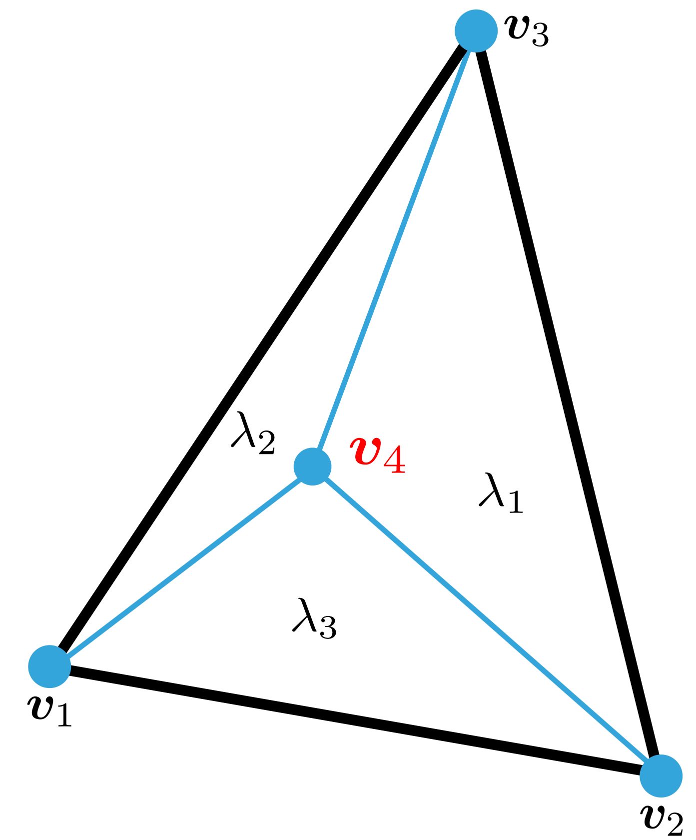


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$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = x$$
$$\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 = y$$

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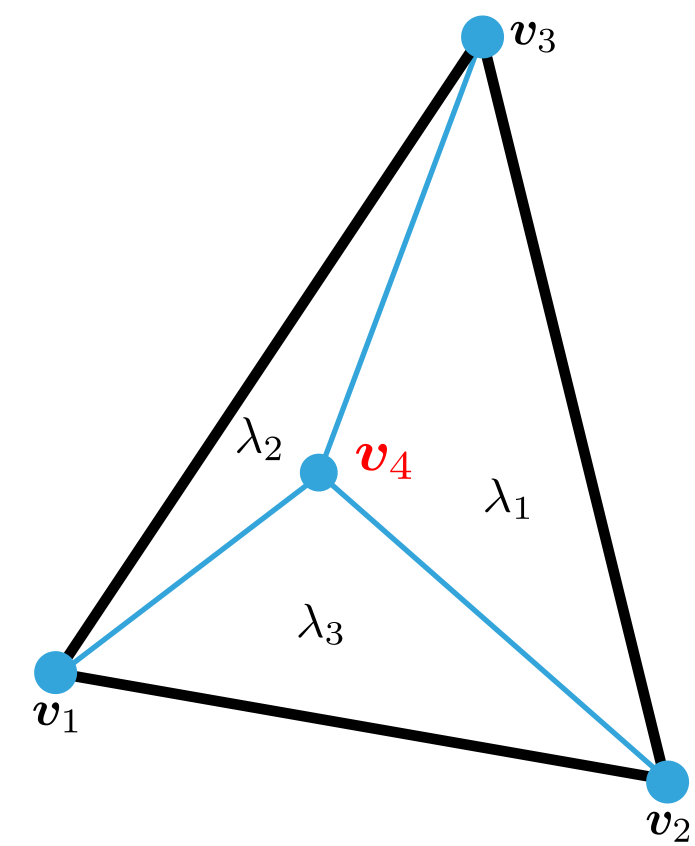



$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

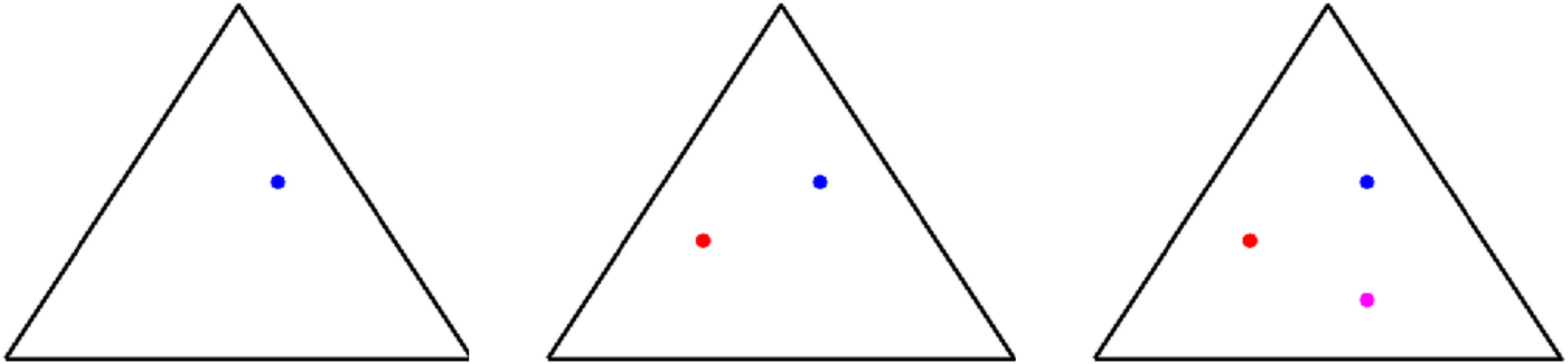
Given $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}, \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$
$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = x$$
$$\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 = y$$

Three equations
Three variables

# SYMMETRY

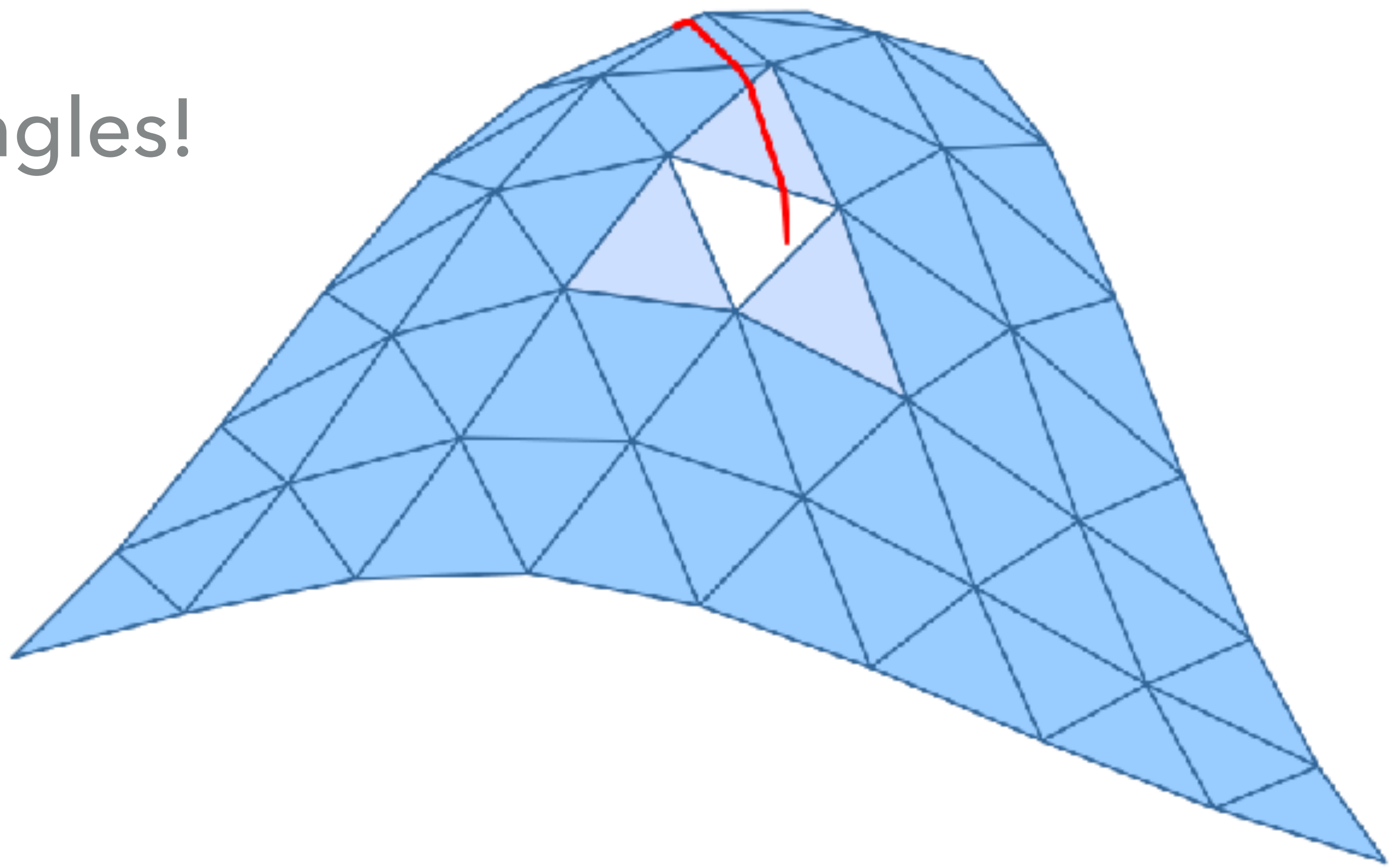
- Rearranging barycentric coordinates also reveals *symmetry*



$$(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{1}{6}, \frac{2}{6}, \frac{3}{6}\right) \quad (\lambda_1, \lambda_2, \lambda_3) = \left(\frac{3}{6}, \frac{1}{6}, \frac{2}{6}\right) \quad (\lambda_1, \lambda_2, \lambda_3) = \left(\frac{2}{6}, \frac{3}{6}, \frac{1}{6}\right)$$

# WHY IS THIS USEFUL?

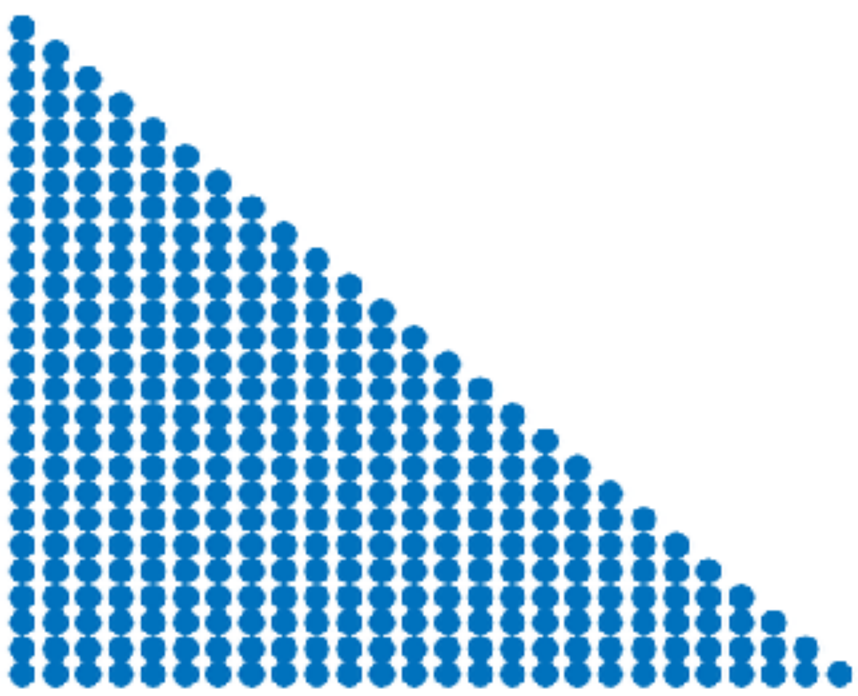
- Can extend to 3D triangles!



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \lambda_2 \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} + \lambda_3 \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

# FILLING IN A SURFACE MADE UP OF TRIANGULAR FACES

- Barycentric coordinates are independent of vertex locations!
- Find barycentric coordinates for points on a reference triangle, then use those same coordinates to find that relative point on a new triangle



Reference triangle

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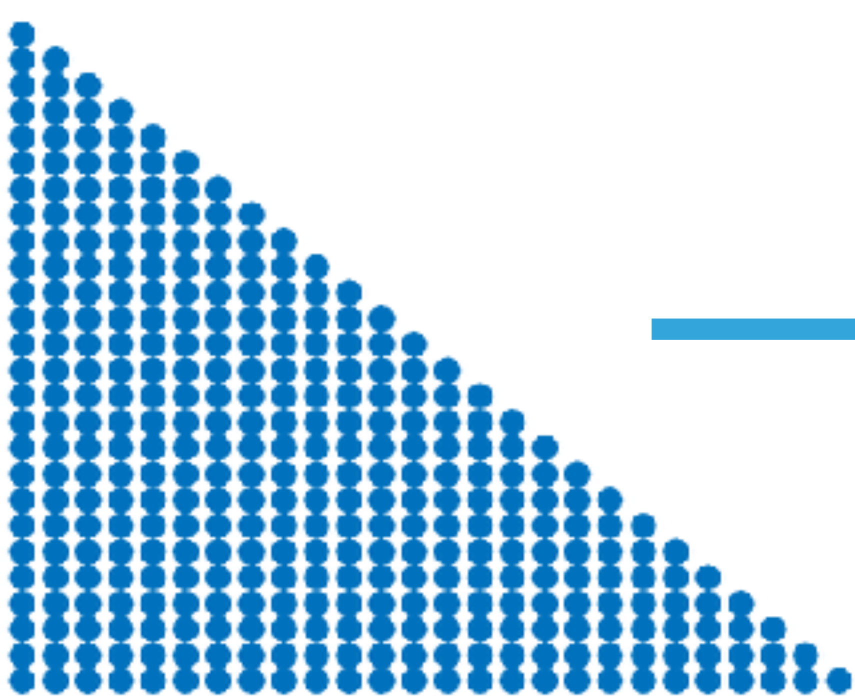
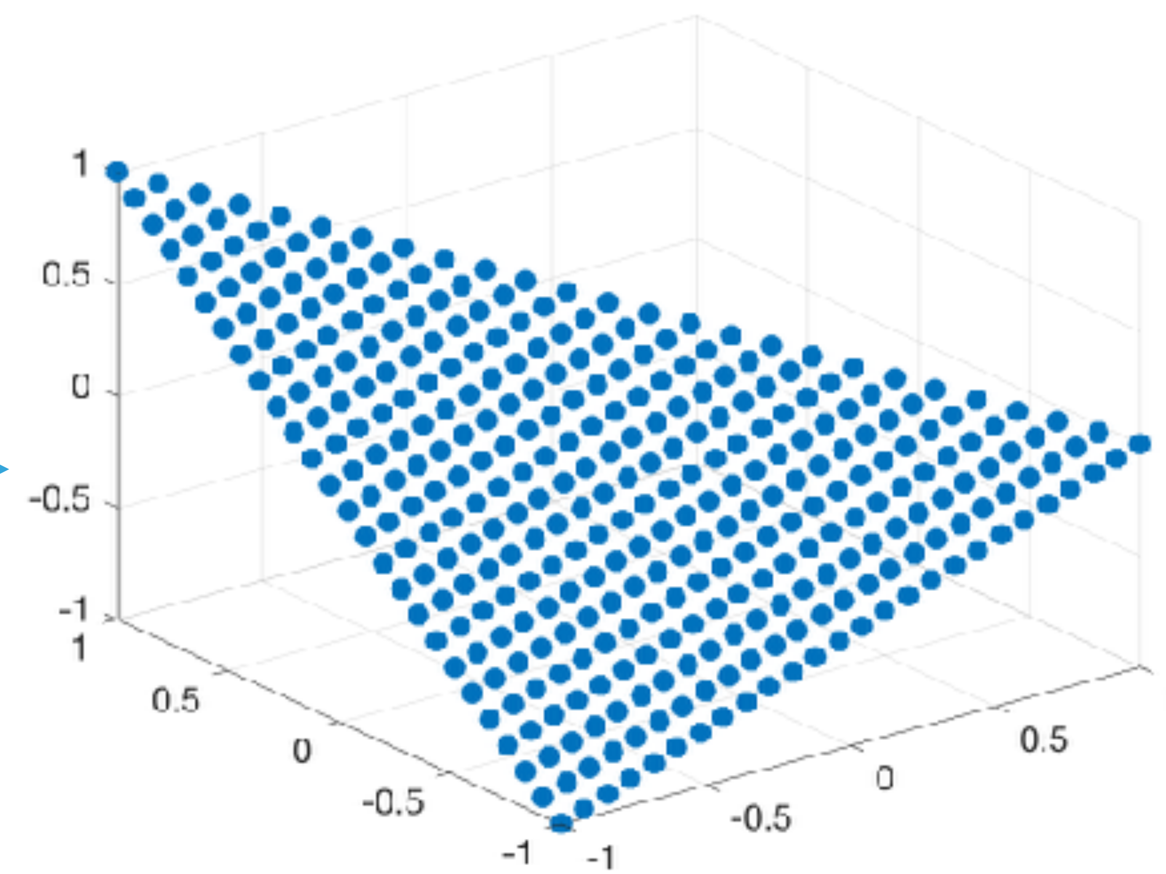
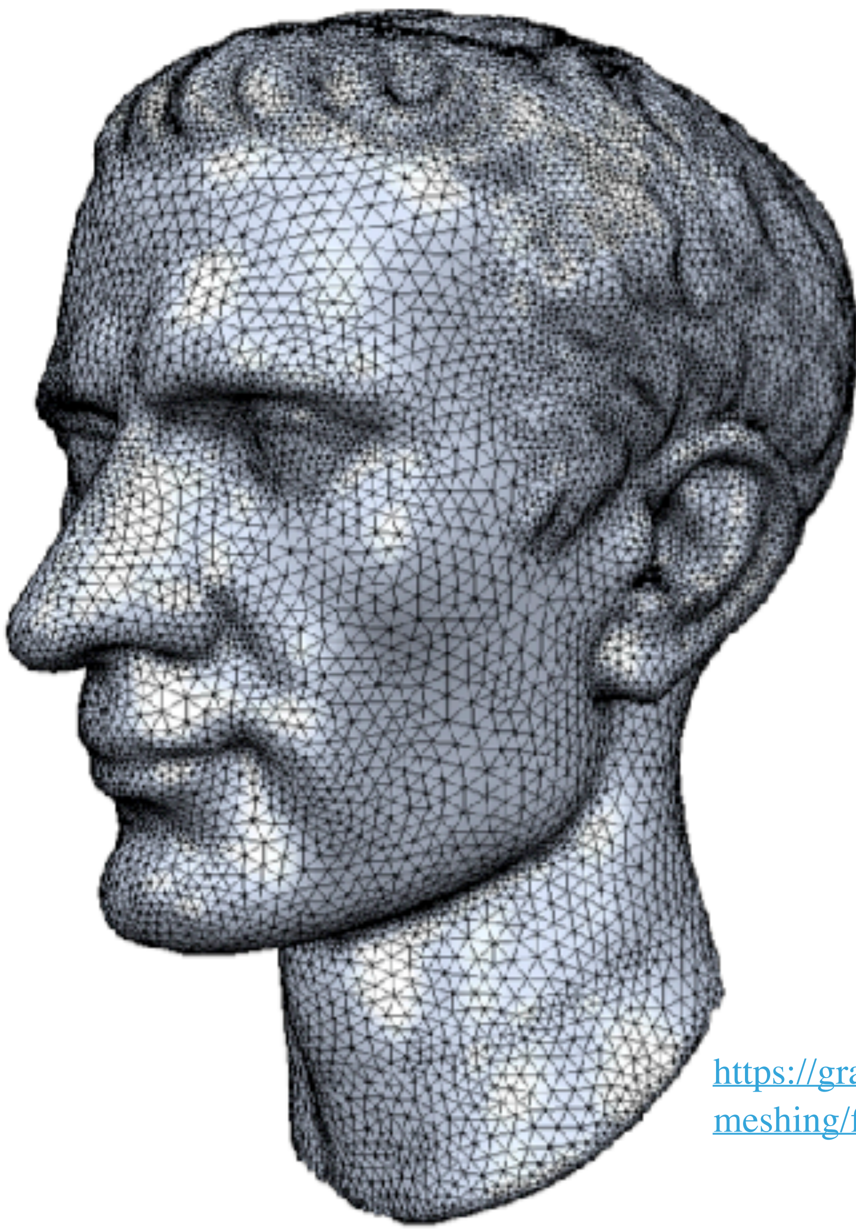


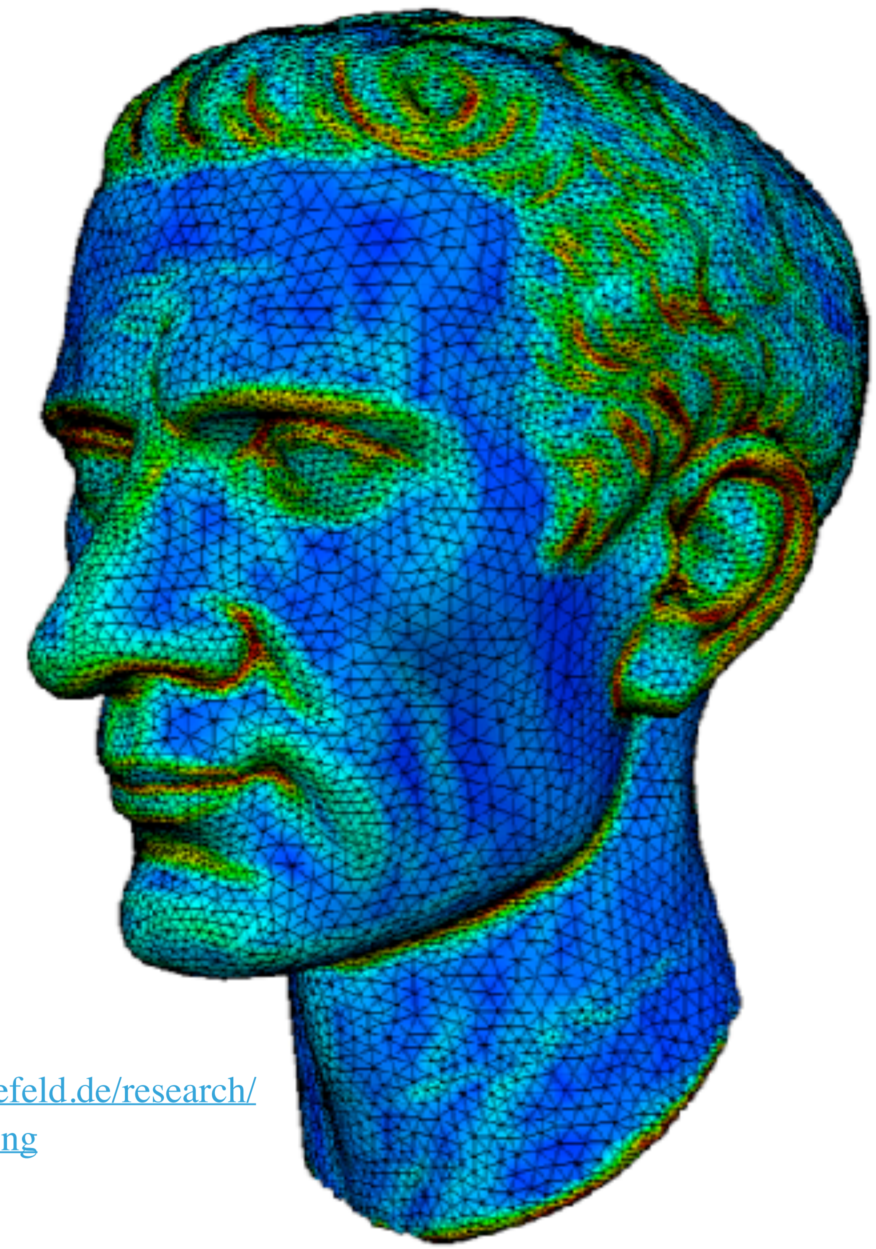
Reference triangle




Physical triangle

# BARYCENTRIC COORDINATES AND TRIANGULAR MESHES

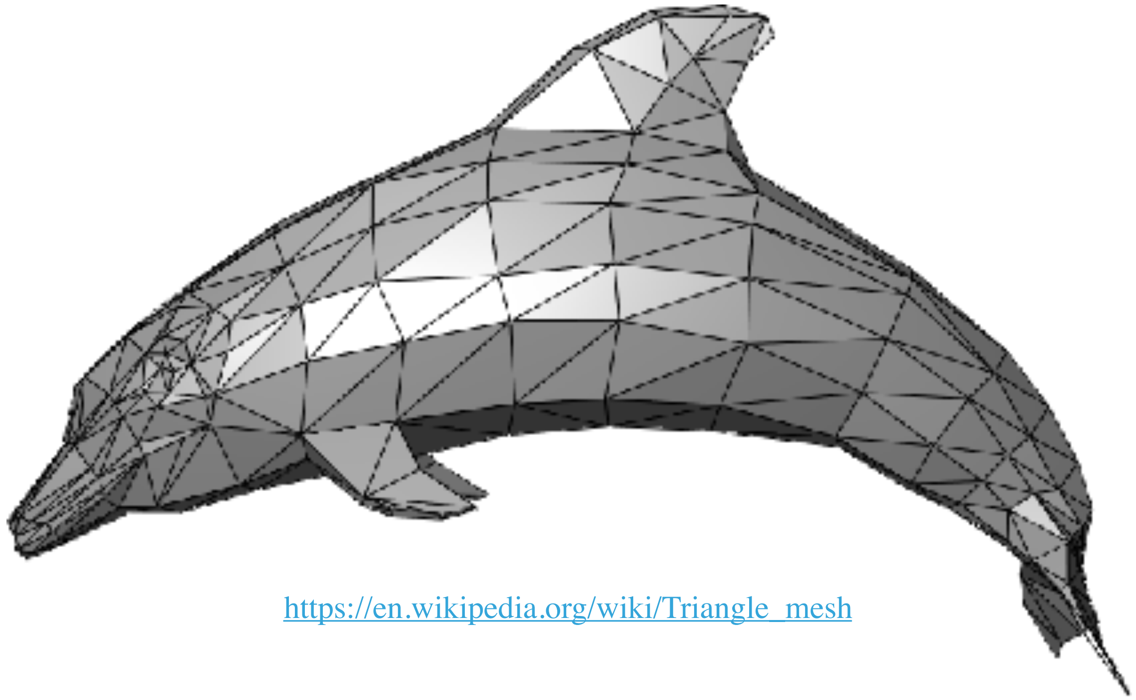




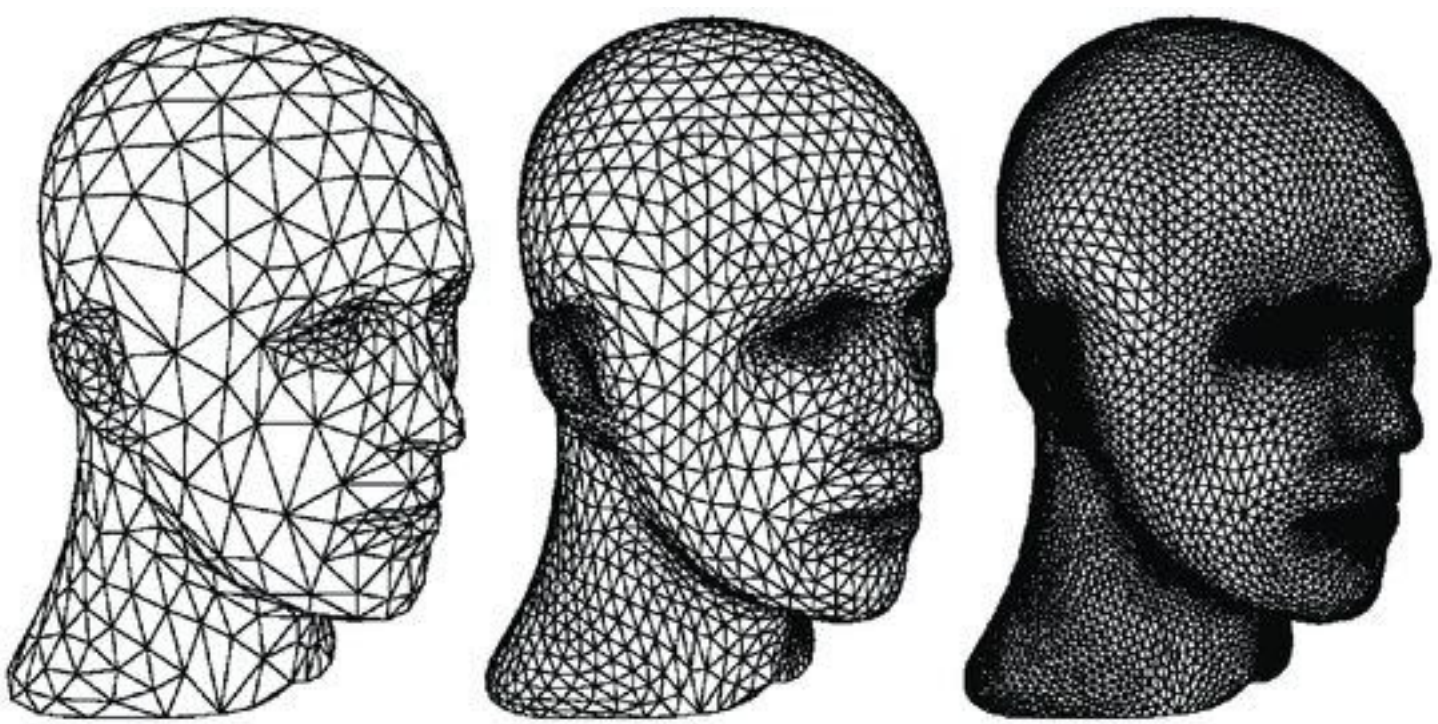
https://graphics.uni-bielefeld.de/research/meshing/files/adaptive.png

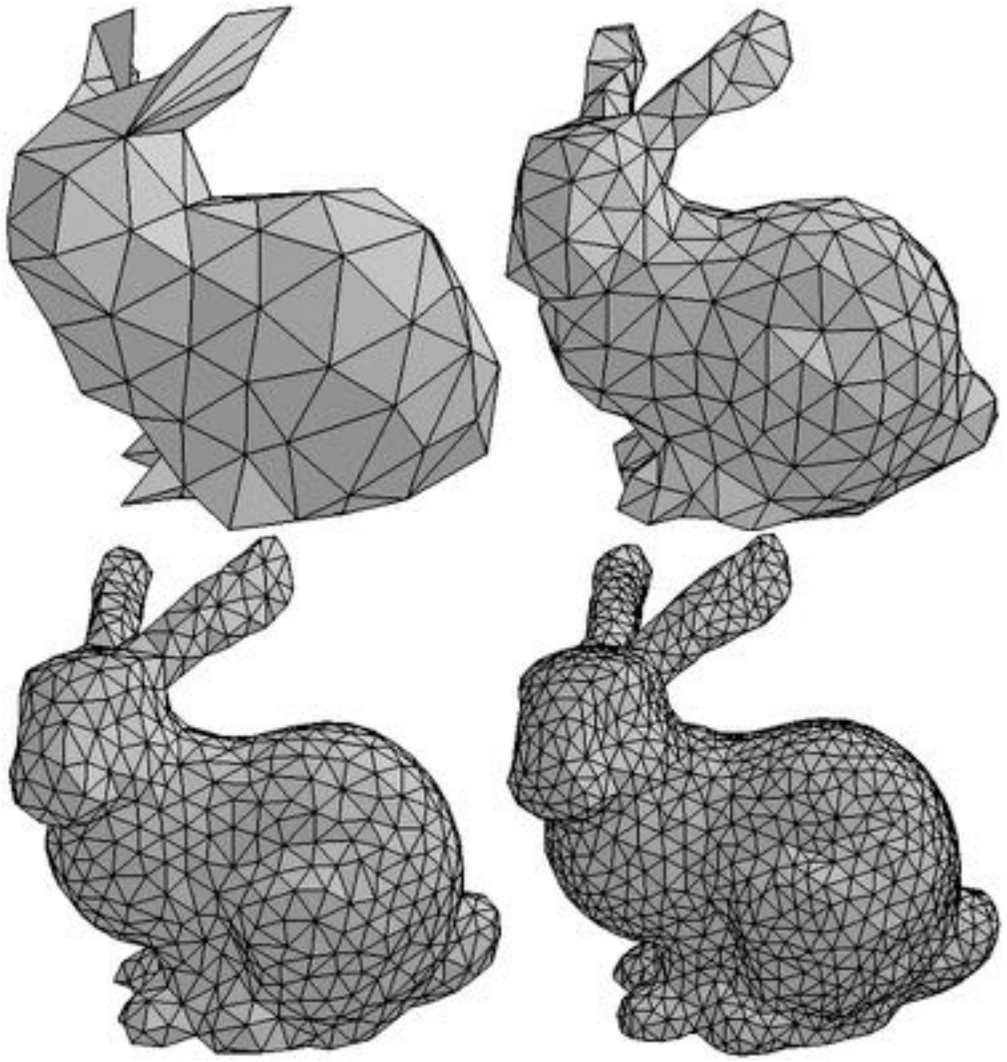
- Triangles are the building blocks of computational geometry
- Examples: computer graphics, CGI, computer-aided design (CAD), numerical simulations)

# TRIANGULAR MESHES: WHO CARES AND WHY?



https://en.wikipedia.org/wiki/Triangle_mesh

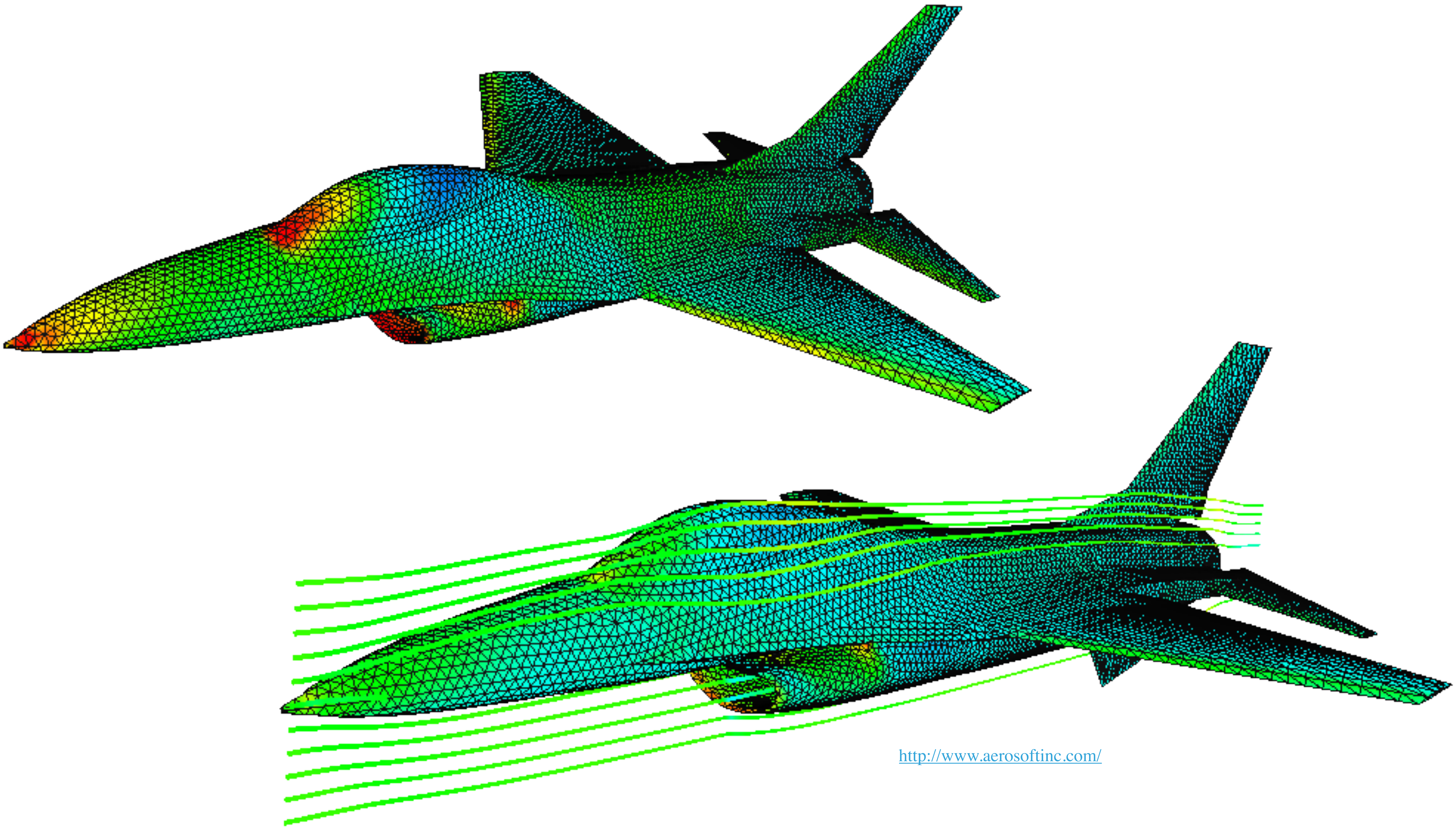




3D Modelling For Programmers

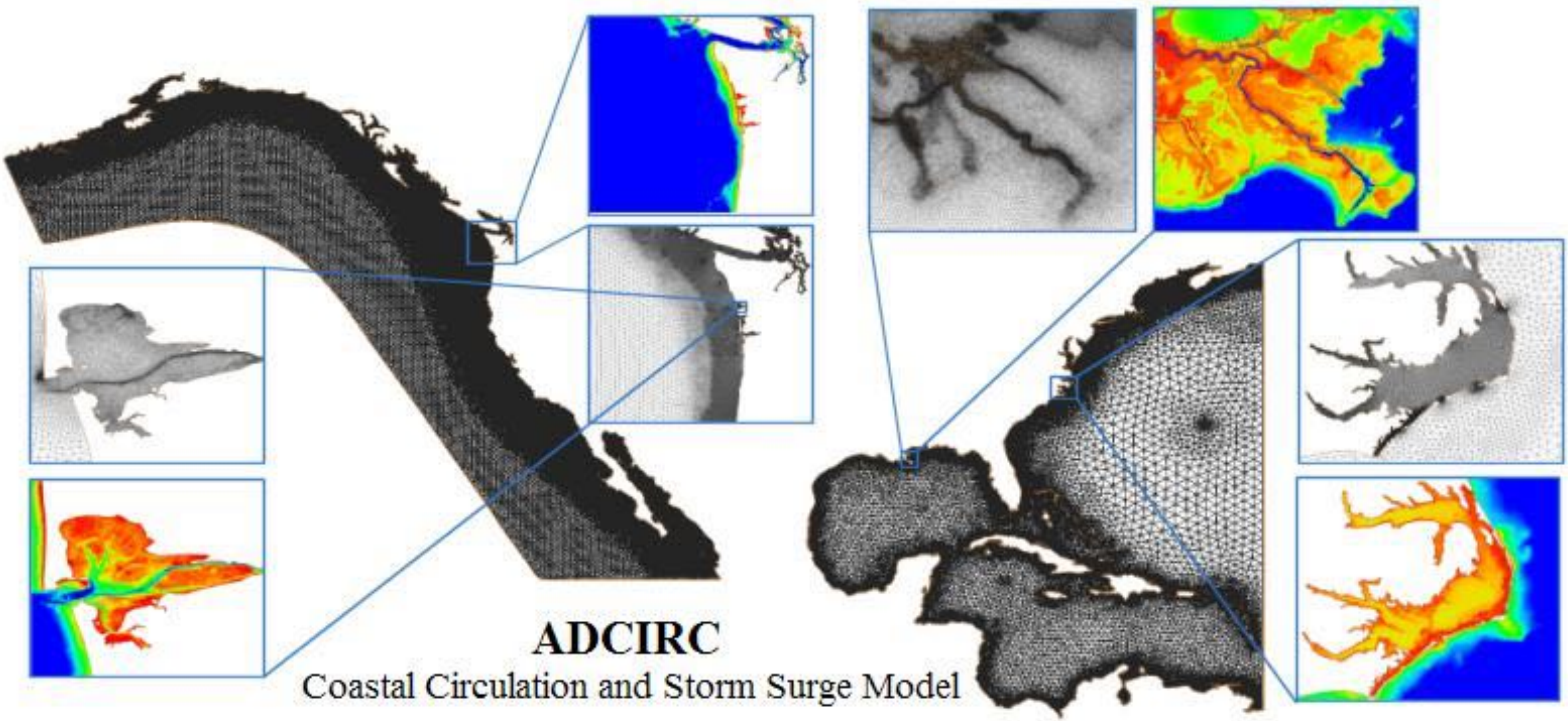
http://pellacini.di.uniroma1.it/teaching/graphics09/lectures/10_SubdivisionSurfaces.pdf

# COMPUTATIONAL SIMULATION AND VISUALIZATION



http://www.aerosoftinc.com/

# NUMERICAL WEATHER PREDICTION



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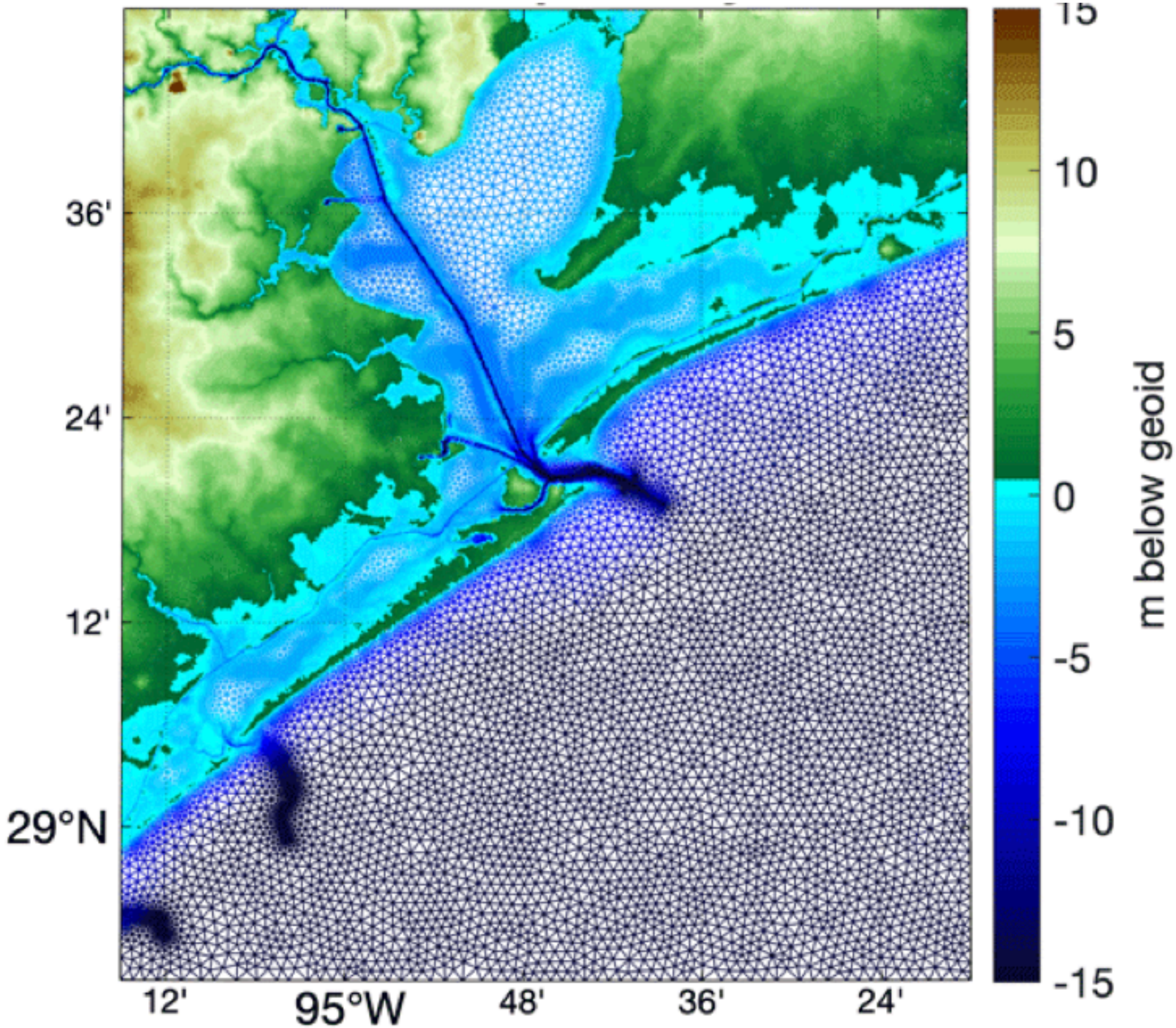


OceanMesh2D 1.0: MATLAB-based software for two-dimensional unstructured mesh generation in coastal ocean modeling

# THANKS FOR LISTENING!