High order entropy stable schemes for the quasi-1D shallow water and compressible Euler equations

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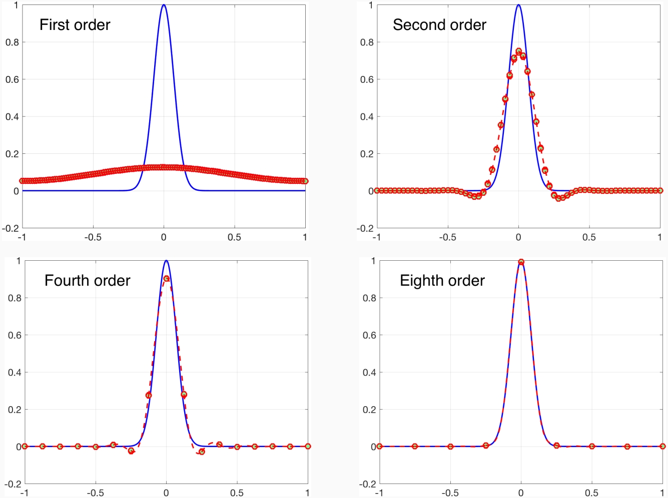
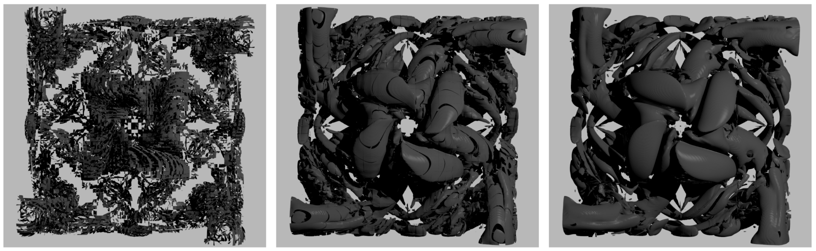
High order methods for time-dependent PDEs



Accurate resolution of propagating vortices and waves.

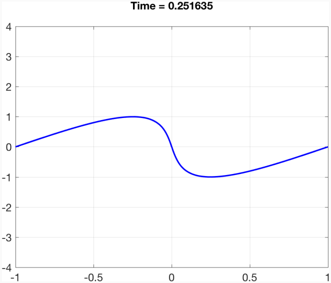
High order methods for time-dependent PDEs



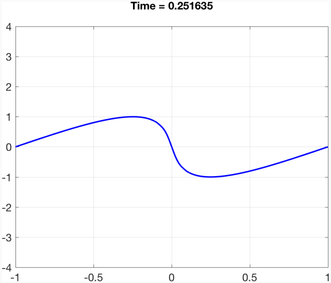
2nd, 4th, and 16th order Taylor-Green vortex. Vorticular structures and acoustic waves are both sensitive to numerical dissipation.

Results from Beck and Gassner (2013).

Why *not* high order methods?



(a) Exact solution



(b) 8th order discontinuous Galerkin (DG)

High order methods blow up for under-resolved solutions of nonlinear conservation laws (e.g., shocks and turbulence).

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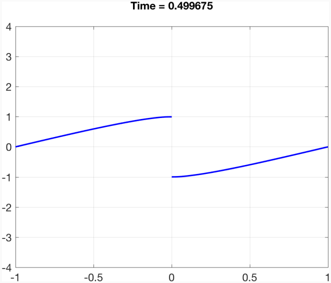

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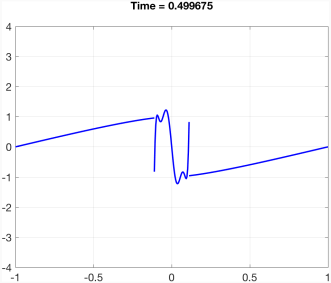

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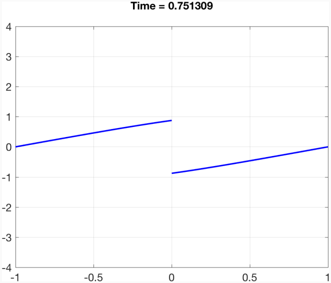

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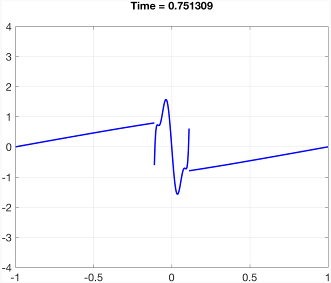

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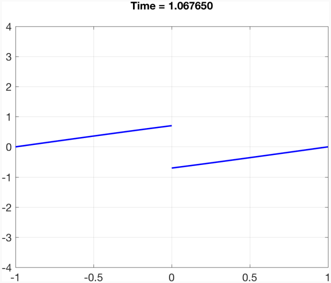

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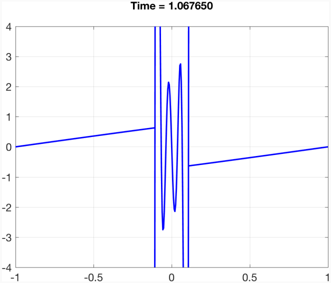

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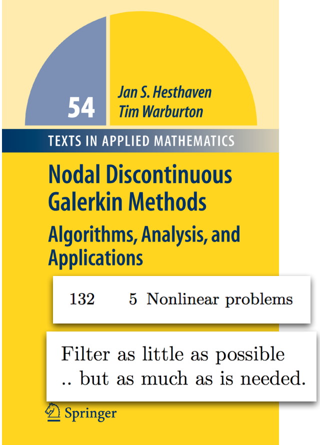
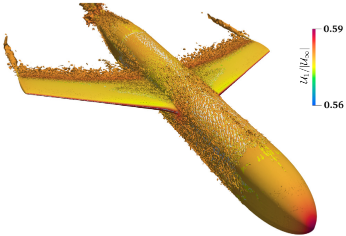
High order entropy stable DG schemes

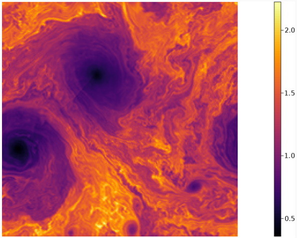


- High order DG needs heuristic stabilization (e.g., artificial viscosity, filtering).
- Entropy stable schemes improve robustness without *no added dissipation*.
- Turns DG into a "good" high order method (though not 100% bulletproof).

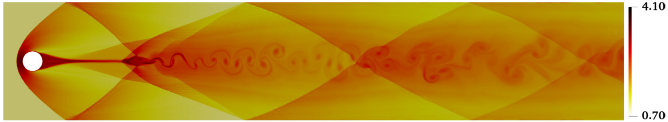
Finite volume methods: Tadmor, Chandrashekar, Ray, Svard, Fjordholm, Mishra, LeFloch, Rohde, ...
High order DGSEM: Fisher, Carpenter, Gassner, Winters, Kopriva, Persson, Pazner, ...
High order simplices: Chen and Shu, Crean, Hicken, Del Rey Fernandez, Zingg, ...

Examples of high order entropy stable DG simulations





(a) Density, ρ.

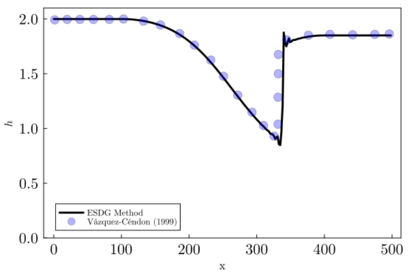


(b) Temperature, $\mathcal{T}$.

All simulations are run without artificial viscosity, filtering, or slope limiting.

Parsani et al. (2021). *High-order accurate entropy-stable discontinuous collocated Galerkin ...*

. . . so why are we talking about 1D equations?



Water height for the quasi-1D shallow water equations.

The quasi-1D shallow water and compressible Euler equations are non-conservative versions of nonlinear conservation laws.

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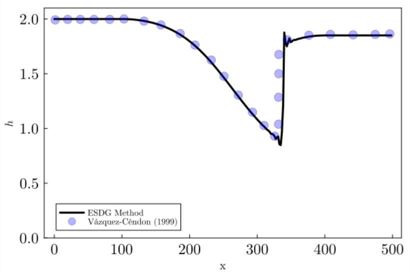

Water height for the quasi-1D shallow water equations.

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Entropy stable nodal DG methods for conservative systems

Entropy stability for conservative systems

- Energy balance for nonlinear conservation laws (Burgers', shallow water, compressible Euler).

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x} = 0.$$

- Continuous entropy inequality: convex entropy function $S(\boldsymbol{u})$, "entropy potential" $\psi(\boldsymbol{u})$, entropy variables $\boldsymbol{v}(\boldsymbol{u})$

$$\int_\Omega \boldsymbol{v}^T \left(\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x} \right) = 0, \qquad \boxed{\boldsymbol{v}(\boldsymbol{u}) = \frac{\partial S}{\partial \boldsymbol{u}}}$$

$$\implies \int_\Omega \frac{\partial S(\boldsymbol{u})}{\partial t} + \left(\boldsymbol{v}^T \boldsymbol{f}(\boldsymbol{u}) - \psi(\boldsymbol{u}) \right) \Big|_{-1}^{1} \leq 0.$$

- The analysis is not as clean for non-conservative systems ...

Entropy conservative finite volume methods

- Finite volume scheme:

$$\frac{\mathrm{d}\mathbf{u}_i}{\mathrm{dt}} + \frac{\boldsymbol{f}(\mathbf{u}_{i+1}, \mathbf{u}_i) - \boldsymbol{f}(\mathbf{u}_{i+1}, \mathbf{u}_i)}{h} = \mathbf{0}.$$

- Take $\boldsymbol{f} = \boldsymbol{f}_{EC}$ to be an entropy conservative numerical flux

$$\begin{aligned} \boldsymbol{f}_{EC}(\boldsymbol{u}, \boldsymbol{u}) &= \boldsymbol{f}(\boldsymbol{u}), && \text{(consistency)} \\ \boldsymbol{f}_{EC}(\boldsymbol{u}, \boldsymbol{v}) &= \boldsymbol{f}_{EC}(\boldsymbol{v}, \boldsymbol{u}), && \text{(symmetry/conservation)} \\ (\boldsymbol{v}_L - \boldsymbol{v}_R)^T \boldsymbol{f}_{EC}(\boldsymbol{u}_L, \boldsymbol{u}_R) &= \psi_L - \psi_R, && \text{(entropy conservation)} \end{aligned}$$

- Can show this numerical scheme conserves entropy

$$\int_\Omega \frac{\partial S(\boldsymbol{u})}{\partial t} \approx \sum_i h \frac{\mathrm{d}S(\mathbf{u}_i)}{\mathrm{dt}} = 0.$$

Tadmor (1987). *The numerical viscosity of entropy stable schemes for systems of conservation laws.*

Example of EC fluxes (compressible Euler equations)

- Define average $\{\{u\}\} = \frac{1}{2}(u_L + u_R)$. In one dimension:

$$f_S^1(\boldsymbol{u}_L, \boldsymbol{u}_R) = \{\{\rho\}\}^{\log} \{\{u\}\}$$
$$f_S^2(\boldsymbol{u}_L, \boldsymbol{u}_R) = \{\{u\}\} f_S^1 + p_{\text{avg}}$$
$$f_S^3(\boldsymbol{u}_L, \boldsymbol{u}_R) = (E_{\text{avg}} + p_{\text{avg}}) \{\{u\}\},$$

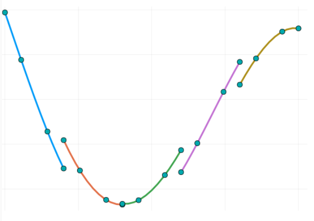
$$p_{\text{avg}} = \frac{\{\{\rho\}\}}{2\{\{\beta\}\}}, \qquad E_{\text{avg}} = \frac{\{\{\rho\}\}^{\log}}{2\{\{\beta\}\}^{\log}(\gamma - 1)} + \frac{1}{2}u_L u_R.$$

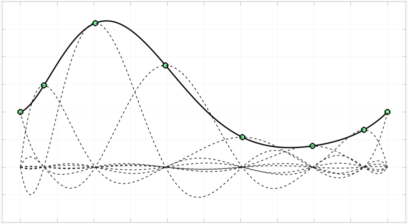
- Non-standard logarithmic mean, "inverse temperature" β

$$\{\{u\}\}^{\log} = \frac{u_L - u_R}{\log u_L - \log u_R}, \qquad \beta = \frac{\rho}{2p}.$$

Chandreshekar (2013), *Kinetic energy preserving and entropy stable finite volume schemes for the compressible Euler and Navier-Stokes equations.*

A brief intro to nodal discontinuous Galerkin methods





- Multiply by nodal (Lagrange) basis $\ell_i(x)$ and integrate

$$\int_{D^k} \left(\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x} \right) \ell_i + \int_{\partial D^k} (\boldsymbol{f}^*(\boldsymbol{u}^+, \boldsymbol{u}^-) - \boldsymbol{f}(\boldsymbol{u}^-)) n \ell_i = 0$$

- The numerical flux $\boldsymbol{f}^*(\boldsymbol{u}^+, \boldsymbol{u}^-) \approx \boldsymbol{f}(\boldsymbol{u})$ enforces boundary conditions and weak continuity across interfaces.
- Nodal (collocation) DG methods: use Gauss-Lobatto quadrature nodes for both interpolation and integration.

Matrix formulation of nodal DG methods

- Map integrals to the reference interval $\widehat{D} = [-1,1]$

$$\int_{\widehat{D}} \left(\frac{h}{2}\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x} \right) \ell_i + \int_{\partial \widehat{D}} (\boldsymbol{f}^*(\boldsymbol{u}^+, \boldsymbol{u}^-) - \boldsymbol{f}(\boldsymbol{u}^-)) n \ell_i = 0$$

- Let $\mathbf{M} = \frac{h}{2}\mathrm{diag}(w_1, \ldots, w_{N+1})$ be a lumped (diagonal) mass matrix and $\mathbf{Q}, \mathbf{B}, \mathbf{E}$ be differentiation and boundary matrices

$$\mathbf{Q}_{ij} = \int_{-1}^{1} \frac{\partial \ell_j}{\partial x}\ell_i, \quad \mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 \end{bmatrix}.$$

Use $\boldsymbol{u}(x,t) = \sum_j \mathbf{u}_j(t)\ell_j(x)$ and $\int_{-1}^{1} \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x}\ell_i \approx \mathbf{Q}\boldsymbol{f}(\mathbf{u})$

$$\boxed{\mathbf{M}\frac{\mathrm{d}\mathbf{u}}{\mathrm{dt}} + \mathbf{Q}\boldsymbol{f}(\mathbf{u}) + \mathbf{E}^T\mathbf{B}\big(\underbrace{\boldsymbol{f}^*\left(\mathbf{u}^+, \mathbf{u}^-\right)}_{\text{interface flux}} - \boldsymbol{f}(\mathbf{u}^-)\big) = \mathbf{0}.}$$

A "flux differencing" formulation

- Key idea: reformulate the DG flux derivative term

$$\int_{-1}^{1} \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x} \ell_i \approx \mathbf{Q}\boldsymbol{f}(\mathbf{u}).$$

- Note that $\mathbf{Q}\mathbf{1} = \mathbf{0}$, so $\sum_j \mathbf{Q}_{ij} = 0$. Thus,

$$(\mathbf{Q}\boldsymbol{f}(\mathbf{u}))_i = \sum_j \mathbf{Q}_{ij}\left(\boldsymbol{f}(\mathbf{u}_j) + \boldsymbol{f}(\mathbf{u}_i)\right) = 2\sum_j \mathbf{Q}_{ij} \underbrace{\frac{\boldsymbol{f}(\mathbf{u}_j) + \boldsymbol{f}(\mathbf{u}_i)}{2}}_{\text{central flux}}$$

- We replace the central flux with an entropy conservative flux

$$2\sum_j \mathbf{Q}_{ij}\boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j) = \left(2\left(\mathbf{Q}\circ\mathbf{F}\right)\mathbf{1}\right)_i, \quad \mathbf{F}_{ij} = \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j).$$

Extension to multiple elements

- An entropy stable nodal DG formulation can be written as:

$$\mathbf{M}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \mathbf{Q}\boldsymbol{f}(\mathbf{u}) + \mathbf{E}^T\mathbf{B}\Big(\underbrace{\boldsymbol{f}^*\left(\mathbf{u}^+, \mathbf{u}^-\right)}_{\text{interface flux}} - \boldsymbol{f}(\mathbf{u}^-)\Big) = \mathbf{0}.$$

- If $\mathbf{Q}$ satisfies the summation-by-parts (SBP) property

$$\mathbf{Q} + \mathbf{Q}^T = \mathbf{E}^T\mathbf{B}\mathbf{E}$$

and if $\boldsymbol{f}^*\left(\mathbf{u}^+, \mathbf{u}\right)$ is entropy stable (e.g., local Lax-Friedrichs flux), a cell entropy inequality holds:

$$\boxed{\int_{D^k} \frac{\partial S(\boldsymbol{u})}{\partial t} + \int_{\partial D^k} \left(\boldsymbol{v}^T \boldsymbol{f}^*(\boldsymbol{u}^+, \boldsymbol{u}^-) - \psi\left(\boldsymbol{u}\right)\right) n \leq 0.}$$

Fisher and Carpenter (2014), Gassner, Winters, and Kopriva (2016), Chen and Shu (2017), etc.

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Entropy stability for systems with non-conservative terms

Entropy stable DG methods

- We have a general framework for entropy stable methods which are (formally) high order accurate.
  - Hybrid meshes, non-conforming interfaces, multi-dimensional and network domains, reduced order models, ...
- Restricted to hyperbolic PDEs in conservation form.
- What if our PDE is not conservative?

Chan (2018). *On discretely entropy conservative and entropy stable discontinuous Galerkin methods.*

Wu, Chan (2020). *Entropy stable discontinuous Galerkin methods for nonlinear conservation laws on networks and multi-dimensional domains*.

Chan, Bencomo, Del Rey Fernandez (2020). *Mortar-based entropy-stable discontinuous Galerkin methods on non-conforming quadrilateral and hexahedral meshes*.

Chan (2020). *Entropy stable reduced order modeling of nonlinear conservation laws*.

Chan (2019). *Skew-symmetric entropy stable modal discontinuous Galerkin formulations*.

Non-conservative systems: shallow water equations

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} = 0$$
$$\frac{\partial hu}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \frac{g}{2}h^2\right) = gh\frac{\partial b}{\partial x}.$$

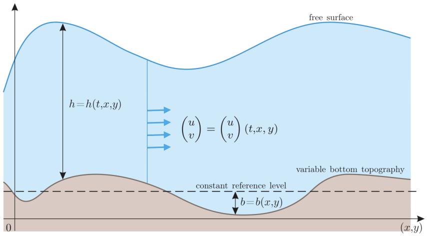


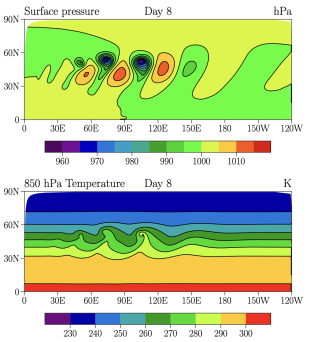
Figure from Bihlo and Popovych (2020)

- Entropy conservation proofs are more complicated and discretization-dependent for non-constant b.
- For discontinuous b, additional DG interface terms needed to retain entropy stability and well-balancedness.

Fjordholm, Mishra, Tadmor (2011). *Well-balanced and energy stable schemes for the shallow water equations with discontinuous topography.*

Non-conservative systems: Euler equations with gravity

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0$$
$$\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}\left(\rho u^2 + p\right) = -\rho \frac{\partial \phi}{\partial x}$$
$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left(u(E+p)\right) = -\rho u \frac{\partial \phi}{\partial x}$$



- Atmospheric flows in weather prediction, climate modeling.
- ϕ is a spatially varying gravitational potential.

Waruszewski et al. (2020). *Entropy stable discontinuous Galerkin methods for balance laws in non-conservative form: Applications to the Euler equations with gravity.*

Non-conservative systems: magnetohydrodynamics (MHD)

$$\begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}_{,t} + \begin{bmatrix} \rho u \\ \rho u^2 + p + \frac{1}{2}\left\|\boldsymbol{B}\right\|^2 - B_1^2 \\ \rho uvw - B_1 B_2 \\ \rho uvw - B_1 B_3 \\ u\widehat{E} - B_1(\boldsymbol{u}\cdot\boldsymbol{B}) + c_h\psi B_1 \\ c_h\psi \\ uB_2 - vB_1 \\ uB_3 - wB_1 \\ c_h B_1 \end{bmatrix}_{,x} = \frac{\partial B_1}{\partial x}\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \boldsymbol{u}\cdot\boldsymbol{B} \\ u \\ v \\ w \\ 0 \end{bmatrix} + \frac{\partial \psi}{\partial x}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u\psi \\ 0 \\ 0 \\ 0 \\ u \end{bmatrix}$$

- Divergence cleaning speed c_h and variable ψ.
- Non-conservative terms necessary for entropy conservation.

Derigs et al (2018). *Ideal GLM-MHD: about the entropy consistent nine-wave magnetic field divergence diminishing ideal magnetohydrodynamics equations.*

What makes non-conservative terms difficult?

Non-conservative terms fall outside of the Tadmor framework:

- Entropy conservation analysis is specific to each non-conservative term and type of discretization.
- General theoretical frameworks typically assume a fully non-conservative nonlinear system.

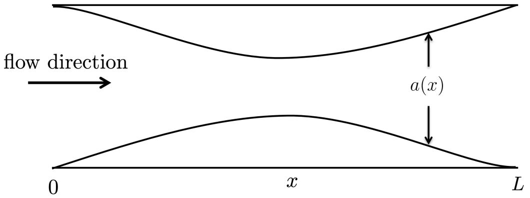
$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{A}(\boldsymbol{u}) \frac{\partial \boldsymbol{u}}{\partial x} = \boldsymbol{0}.$$

Implementations based on theoretical frameworks are more complicated than implementations for conservative systems.

Castro, Fjordholm, Mishra, Parés (2013). *Entropy conservative and entropy stable schemes for nonconservative hyperbolic systems.*

Renac, Florent (2019). *Entropy stable DGSEM for nonlinear hyperbolic systems in nonconservative form with application to two-phase flows.*

Quasi-1D versions of nonlinear conservation laws



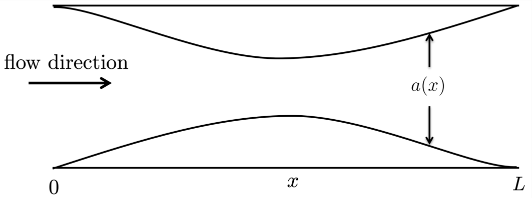
Quasi-1D shallow water equations

$$\frac{\partial}{\partial t}\begin{bmatrix} ah \\ ahu \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} ahu \\ ahu^2 \end{bmatrix} + \begin{bmatrix} 0 \\ agh\frac{\partial}{\partial x}(h+b) \end{bmatrix} = 0.$$

Quasi-1D compressible Euler equations

$$\frac{\partial}{\partial t}\begin{bmatrix} a\rho \\ a\rho u \\ aE \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} a\rho u \\ a\rho u^2 \\ au(E+p) \end{bmatrix} + a\frac{\partial}{\partial x}\begin{bmatrix} 0 \\ p \\ 0 \end{bmatrix} = 0.$$

Continuous entropy analysis



For both quasi-1D shallow water and compressible Euler:

- An appropriate convex entropy is the scaled entropy $a(x)S(\boldsymbol{u})$
- Under this entropy, the entropy variables for the quasi-1D equations are identical to the standard 1D entropy variables.
- Sufficiently regular solutions satisfy a conservation of entropy.

Chan, Shukla, Wu, Liu, Nalluri (2023). *High order entropy stable schemes for the quasi-one-dimensional shallow water and compressible Euler equations.*

The symmetric Tadmor condition for conservative systems

If we assume $\boldsymbol{f}_{EC}\left(\boldsymbol{u}_L, \boldsymbol{u}_R\right) = \boldsymbol{f}_{EC}\left(\boldsymbol{u}_R, \boldsymbol{u}_L\right)$, Tadmor's condition:

$$\boxed{\left(\boldsymbol{v}_L - \boldsymbol{v}_R\right)^T \boldsymbol{f}_{EC}\left(\boldsymbol{u}_L, \boldsymbol{u}_R\right) = \psi_L - \psi_R.}$$

Main step in proof of entropy stability (assumes $\mathbf{Q} = -\mathbf{Q}^T$):

$$
\begin{aligned}
\mathbf{v}^T 2\left(\mathbf{Q} \circ \mathbf{F}\right)\mathbf{1} &= 2\sum_{ij} \mathbf{Q}_{ij}\mathbf{v}_i^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j) \\
&= \sum_{ij} \mathbf{Q}_{ij}\mathbf{v}_i^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j) - \mathbf{Q}_{ji}\mathbf{v}_i^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j) \\
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&= \sum_{ij} \mathbf{Q}_{ij}\underbrace{\left(\mathbf{v}_i - \mathbf{v}_j\right)^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j)}_{\psi(\mathbf{u}_i) - \psi(\mathbf{u}_j)}.
\end{aligned}
$$

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&= \sum_{ij} \mathbf{Q}_{ij}\mathbf{v}_i^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j) - \mathbf{Q}_{ij}\mathbf{v}_j^T \boldsymbol{f}_{EC}(\mathbf{u}_j, \mathbf{u}_i) \\
&= \sum_{ij} \mathbf{Q}_{ij} \underbrace{\left(\mathbf{v}_i - \mathbf{v}_j\right)^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j)}_{\psi(\mathbf{u}_i) - \psi(\mathbf{u}_j)}.
\end{aligned}$$

The symmetric Tadmor condition for conservative systems

If we assume $\boldsymbol{f}_{EC}\left(\boldsymbol{u}_L, \boldsymbol{u}_R\right) = \boldsymbol{f}_{EC}\left(\boldsymbol{u}_R, \boldsymbol{u}_L\right)$, Tadmor's condition:

$$\boxed{\left(\boldsymbol{v}_L - \boldsymbol{v}_R\right)^T \boldsymbol{f}_{EC}\left(\boldsymbol{u}_L, \boldsymbol{u}_R\right) = \psi_L - \psi_R.}$$

Main step in proof of entropy stability (assumes $\mathbf{Q} = -\mathbf{Q}^T$):

$$\begin{aligned}
\mathbf{v}^T 2\left(\mathbf{Q} \circ \mathbf{F}\right)\mathbf{1} &= 2\sum_{ij} \mathbf{Q}_{ij}\mathbf{v}_i^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j) \\
&= \sum_{ij} \mathbf{Q}_{ij}\mathbf{v}_i^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j) - \mathbf{Q}_{ji}\mathbf{v}_i^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j) \\
&= \sum_{ij} \mathbf{Q}_{ij}\mathbf{v}_i^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j) - \mathbf{Q}_{ij}\mathbf{v}_j^T \boldsymbol{f}_{EC}(\mathbf{u}_j, \mathbf{u}_i) \\
&= \sum_{ij} \mathbf{Q}_{ij} \underbrace{\left(\mathbf{v}_i - \mathbf{v}_j\right)^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j)}_{\psi(\mathbf{u}_i) - \psi(\mathbf{u}_j)}.
\end{aligned}$$

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&= \sum_{ij} \mathbf{Q}_{ij}\mathbf{v}_i^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j) - \mathbf{Q}_{ji}\mathbf{v}_i^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j) \\
&= \sum_{ij} \mathbf{Q}_{ij}\mathbf{v}_i^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j) - \mathbf{Q}_{ij}\mathbf{v}_j^T \boldsymbol{f}_{EC}(\mathbf{u}_j, \mathbf{u}_i) \\
&= \sum_{ij} \mathbf{Q}_{ij} \underbrace{\left(\mathbf{v}_i - \mathbf{v}_j\right)^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j)}_{\psi(\mathbf{u}_i) - \psi(\mathbf{u}_j)}.
\end{aligned}$$

A non-symmetric version of Tadmor's condition

If we do not assume symmetry of $\boldsymbol{f}_{EC}\left(\boldsymbol{u}_L, \boldsymbol{u}_R\right)$:

$$\boxed{\boldsymbol{v}_L^T \boldsymbol{f}_{EC}\left(\boldsymbol{u}_L, \boldsymbol{u}_R\right) - \boldsymbol{v}_R^T \boldsymbol{f}_{EC}\left(\boldsymbol{u}_R, \boldsymbol{u}_L\right) = \psi_L - \psi_R.}$$

Main step in proof of entropy stability (assumes $\mathbf{Q} = -\mathbf{Q}^T$):

$$\begin{aligned}
\mathbf{v}^T 2\left(\mathbf{Q} \circ \mathbf{F}\right)\mathbf{1} &= 2\sum_{ij} \mathbf{Q}_{ij} \mathbf{v}_i^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j) \\
&= \sum_{ij} \mathbf{Q}_{ij} \mathbf{v}_i^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j) - \mathbf{Q}_{ji} \mathbf{v}_i^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j) \\
&= \sum_{ij} \mathbf{Q}_{ij} \mathbf{v}_i^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j) - \mathbf{Q}_{ij} \mathbf{v}_j^T \boldsymbol{f}_{EC}(\mathbf{u}_j, \mathbf{u}_i) \\
&= \sum_{ij} \mathbf{Q}_{ij} \underbrace{\left(\mathbf{v}_i^T \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j) - \mathbf{v}_j^T \boldsymbol{f}_{EC}(\mathbf{u}_j, \mathbf{u}_i)\right)}_{\psi(\mathbf{u}_i) - \psi(\mathbf{u}_j)}.
\end{aligned}$$

What do you give up without symmetry?

- Proof of high order consistency for “flux differencing” valid only for symmetric (e.g., central-like) finite volume fluxes.

$$((\mathbf{Q} \circ \mathbf{F})\,\mathbf{1})_i \approx \int_D \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x} \ell_i(x), \qquad \mathbf{F}_{ij} = \boldsymbol{f}_{EC}(\mathbf{u}_i, \mathbf{u}_j).$$

  where $\mathbf{Q}$ is a degree N differentiation matrix and $\boldsymbol{u}$ is a degree N polynomial approximation.

- Non-symmetric finite volume flux $\boldsymbol{f}_{EC}(\boldsymbol{u}_L, \boldsymbol{u}_R)$: no guarantee that “flux differencing” is high order accurate!

- Lucky for us: non-symmetric terms typically easy to analyze.

Crean, Jared, et al (2018). *Entropy-stable summation-by-parts discretization of the Euler equations on general curved elements.*

Consistency of non-conservative terms

Suppose that $a_i = a(x_i)$, $u_i = u(x_i)$, and $\mathbf{F}_{ij} = a_i u_j$.

$$((\mathbf{Q} \circ \mathbf{F})\,\mathbf{1})_i = \sum_j \mathbf{Q}_{ij} a_i u_j = a_i \sum_j \mathbf{Q}_{ij} u_j \approx \int a \frac{\partial u}{\partial x} \ell_i.$$

Our new entropy conservative fluxes for the quasi-1D shallow water equations are:

$$f_h = \{\{ahu\}\}$$

$$f_{hu} = \{\{ahu\}\}\,\{\{u\}\} + \boxed{\frac{g}{2} a_L h_L (h_R + b_R)}.$$

The non-symmetric term (boxed) is a consistent, high order accurate approximation of $\int ah \frac{\partial}{\partial x}(h+b)\ell_i$.

Suppose that $a_i = a(x_i)$, $u_i = u(x_i)$, and $\mathbf{F}_{ij} = a_i u_j$.

$$((\mathbf{Q} \circ \mathbf{F})\, \mathbf{1})_i = \sum_j \mathbf{Q}_{ij} a_i u_j = a_i \sum_j \mathbf{Q}_{ij} u_j \approx \int a \frac{\partial u}{\partial x} \ell_i.$$

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The non-symmetric term (boxed) is a consistent, high order accurate approximation of $\int ah \frac{\partial}{\partial x}(h+b)\ell_i$.

New entropy conservative flux: quasi-1D compressible Euler

Quasi-1D compressible Euler equations:

$$
\begin{aligned}
f_\rho &= \{\{\rho\}\}_{\log} \{\{au\}\}, \\
f_{\rho u} &= \{\{\rho\}\}_{\log} \{\{au\}\} \{\{u\}\} + \boxed{a_L \{\{p\}\}}, \\
f_E &= \frac{1}{2} \{\{\rho\}\}_{\log} \{\{au\}\} ((u \cdot u)) + \\
&\quad \frac{1}{\gamma - 1} \{\{\rho\}\}_{\log} \{\{\rho/p\}\}_{\log}^{-1} \{\{au\}\} + ((p \cdot au)),
\end{aligned}
$$

with logarithmic and product means

$$
\{\{\rho\}\}_{\log} := \frac{[\![\rho]\!]}{[\![\log\ \rho]\!]} = \frac{\rho_L - \rho_R}{\log(\rho_L) - \log(\rho_R)}, \quad ((u \cdot v)) := \frac{u_L v_R + u_R v_L}{2}.
$$

Ranocha (2018). *Comparison of some entropy conservative numerical fluxes for the Euler equations.*

Convergence study: quasi-1D shallow water

	$N=1$		$N=2$		$N=3$		$N=4$	
K	L^2 error	Rate	L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
2	1.43	-	1.22	-	7.05×10^{-1}	-	4.07×10^{-1}	-
4	1.26	0.19	3.0×10^{-1}	2.05	1.18×10^{-1}	2.61	3.28×10^{-2}	3.63
8	5.14×10^{-1}	1.29	1.00×10^{-1}	1.56	1.48×10^{-2}	2.97	1.89×10^{-3}	4.12
16	2.01×10^{-1}	1.35	1.58×10^{-2}	2.67	6.90×10^{-4}	4.41	1.82×10^{-4}	3.37
32	7.21×10^{-2}	1.48	2.53×10^{-3}	2.64	7.88×10^{-5}	3.12	6.98×10^{-6}	4.71

(a) Reference solution

	$N=1$		$N=2$		$N=3$	
K	L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
16	4.80×10^{-1}	-	8.46×10^{2}	-	8.53×10^{-3}	-
32	1.30×10^{-1}	1.88	9.89×10^{-3}	3.10	4.73×10^{-4}	4.17
64	3.44×10^{-2}	1.92	1.23×10^{-3}	3.00	2.96×10^{-5}	4.00
128	8.95×10^{-3}	1.94	1.54×10^{-4}	2.99	1.87×10^{-6}	3.99
256	2.28×10^{-3}	1.97	1.93×10^{-5}	3.00	1.17×10^{-7}	4.00

	$N=4$		$N=5$	
K	L^2 error	Rate	L^2 error	Rate
16	1.00×10^{-3}	-	1.39×10^{-4}	-
32	3.45×10^{-5}	4.86	1.82×10^{-6}	6.25
64	1.09×10^{-6}	4.98	2.84×10^{-8}	6.00
128	3.39×10^{-8}	5.01	4.48×10^{-10}	5.99
256	1.05×10^{-9}	5.01	7.04×10^{-11}	5.99

(b) Manufactured solution

Convergence study: quasi-1D compressible Euler

	$N=1$		$N=2$		$N=3$		$N=4$	
K	L^2 error	Rate	L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
2	1.116×10^{-1}	-	1.0274×10^{-1}	-	6.571×10^{-2}	-	3.429×10^{-2}	-
4	1.061×10^{-1}	0.07	4.575×10^{-2}	1.17	1.666×10^{-2}	1.98	4.349×10^{-3}	2.98
8	5.049×10^{-2}	1.07	1.475×10^{-2}	1.63	2.089×10^{-3}	3.0	2.604×10^{-4}	4.06
16	2.001×10^{-2}	1.34	2.481×10^{-3}	2.57	1.416×10^{-4}	3.88	8.918×10^{-6}	4.87
32	7.111×10^{-3}	1.49	3.201×10^{-4}	2.95	9.006×10^{-6}	3.97	3.001×10^{-6}	4.89

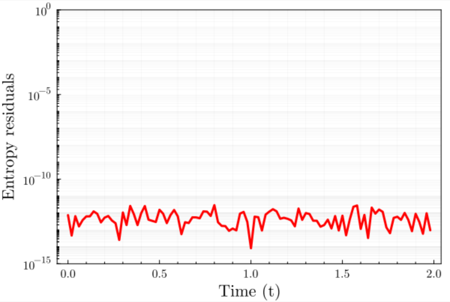
(a) Reference solution

	$N=1$		$N=2$		$N=3$	
N_{elem}	L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
5	8.514×10^{-1}	-	8.056×10^{-1}	-	3.518×10^{-2}	-
10	3.107×10^{-1}	1.45	2.354×10^{-1}	2.844	2.307×10^{-3}	3.930
20	8.833×10^{-2}	1.81	3.277×10^{-2}	2.731	3.843×10^{-4}	3.942
40	2.280×10^{-2}	1.95	4.936×10^{-3}	2.537	2.619×10^{-5}	3.715
80	5.712×10^{-3}	1.97	8.505×10^{-4}	2.392	7.910×10^{-7}	3.853

	$N=4$		$N=5$	
N_{elem}	L^2 error	Rate	L^2 error	Rate
5	5.539×10^{-3}	-	3.961×10^{-2}	-
10	2.100×10^{-4}	4.72	2.749×10^{-3}	5.64
20	1.029×10^{-5}	4.35	5.790×10^{-5}	5.85
40	5.083×10^{-7}	4.34	1.169×10^{-6}	5.95
80	2.599×10^{-8}	4.29	1.981×10^{-8}	6.00

(b) Manufactured solution

Entropy conservation and well-balancedness

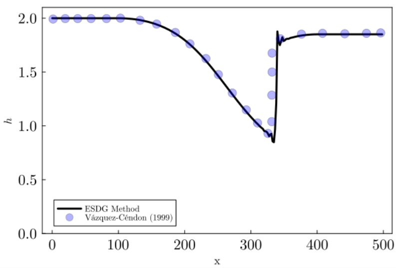


Entropy residual for an entropy conservative simulation of the quasi-1D compressible Euler equations with discontinuous initial condition.

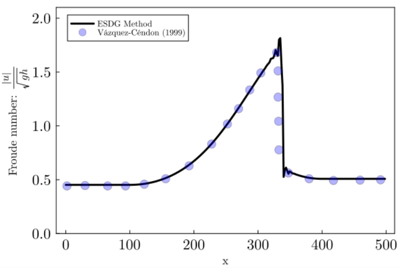
	L^1 error	L^∞ error
Continuous b and a	9.19×10^{-15}	2.01×10^{-13}
Discontinuous b and a	1.46×10^{-14}	1.65×10^{-16}

Errors for the well-balanced test for the quasi-1D shallow water equations.

Quasi-1D shallow water: converging-diverging channel



(a) h

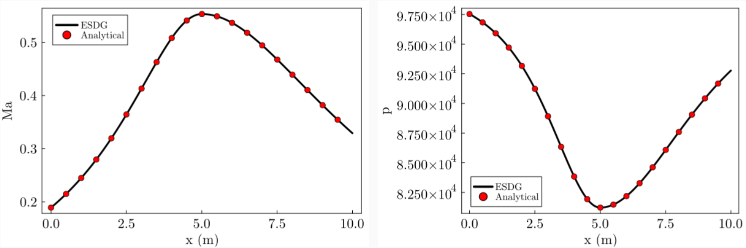


(b) Froude number

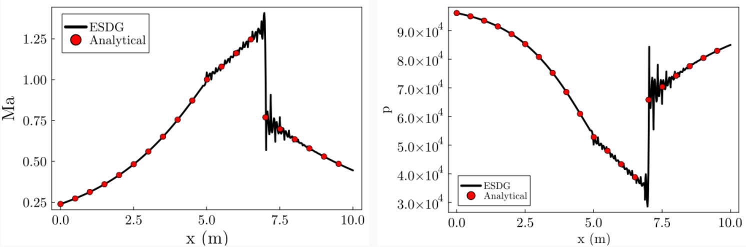
Water height and Froude number for the converging-diverging channel problem.

Vázquez-Cendón (1999). *Improved treatment of source terms in upwind schemes for the shallow water equations in channels with irregular geometry.*

Quasi-1D compressible Euler: convergent-divergent nozzle



Mach number and pressure for subsonic flow through a nozzle.



Mach number and pressure for transonic flow through a nozzle.

Conclusions and acknowledgements

- A non-symmetric version of the Tadmor condition simplifies the analysis of non-conservative systems.
- Application to the quasi-1D shallow water and compressible Euler equations: entropy conservation and well-balancedness for arbitrary channel widths and bathymetry.

This work is supported by DMS-1943186 and DMS-2231482.

Thank you! Questions?

Chan, Shukla, Wu, Liu, Nalluri (2023). *High order entropy stable schemes for the quasi-one-dimensional shallow water and compressible Euler equations.*